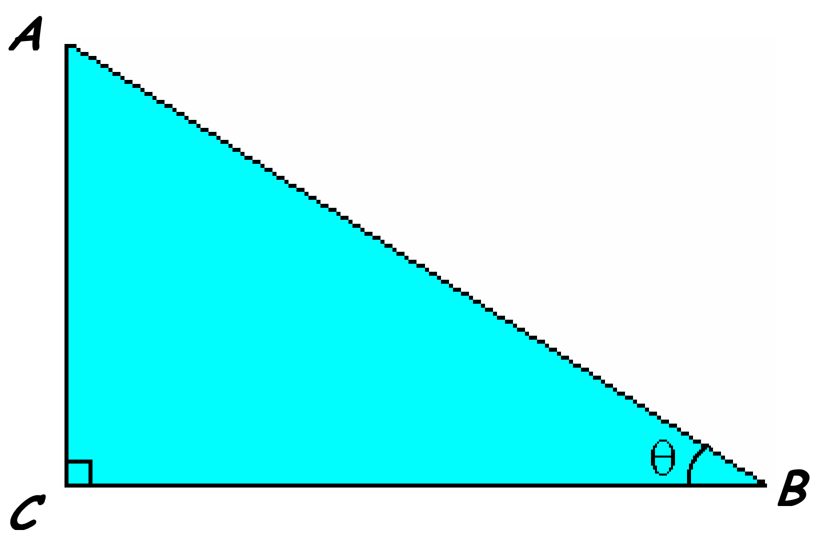
# Lesson 29

## Glencoe Geometry Chapter 8.2

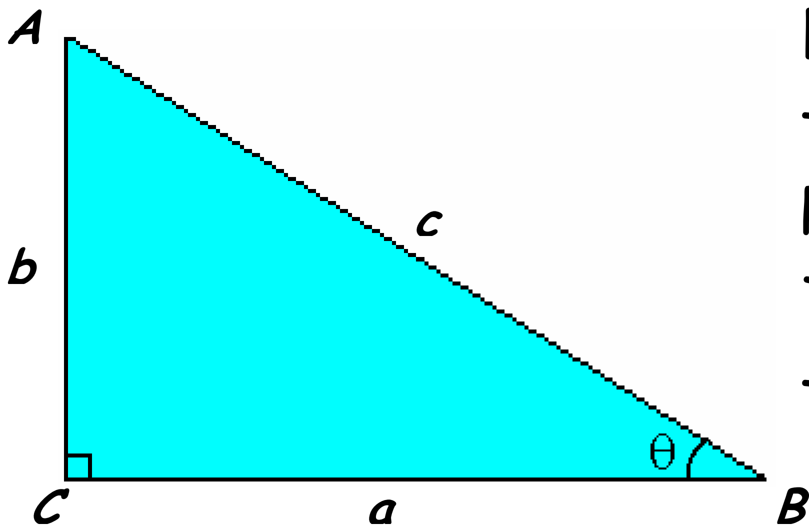
# Special Right Triangles

Today we will look at two special right triangles, their side lengths, and their angle measure. These two triangles are not only very important in geometry, but the form the basis of the study of coordinate geometry and trigonometry.

Remember a right triangle is a three-sided polygon with exactly one angle of 90°. The other two angles are complementary, meaning the sum of their measures is also 90°.



We can label the vertices as capital letters, and the line segments opposite the angles as the corresponding lower-case letters.




For all right triangles, the Pythagorean Theorem relates the three sides:

$$a^2 + b^2 = c^2$$

We donate an angle (and its measure) by using $\theta$, the capital Greek letter "theta."

When specifically referring to a one of the two non-right angles, like above, we call that angle the reference angle.

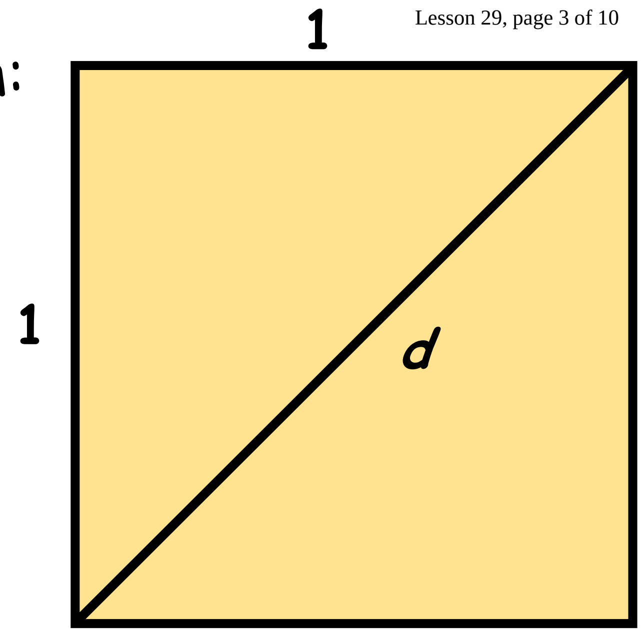
Let's start our study by looking at a unit square (each side of one unit in length) and its diagonal, ***d***.

By the Pythagorean theorem:

$$d^2 = 1^2 + 1^2$$

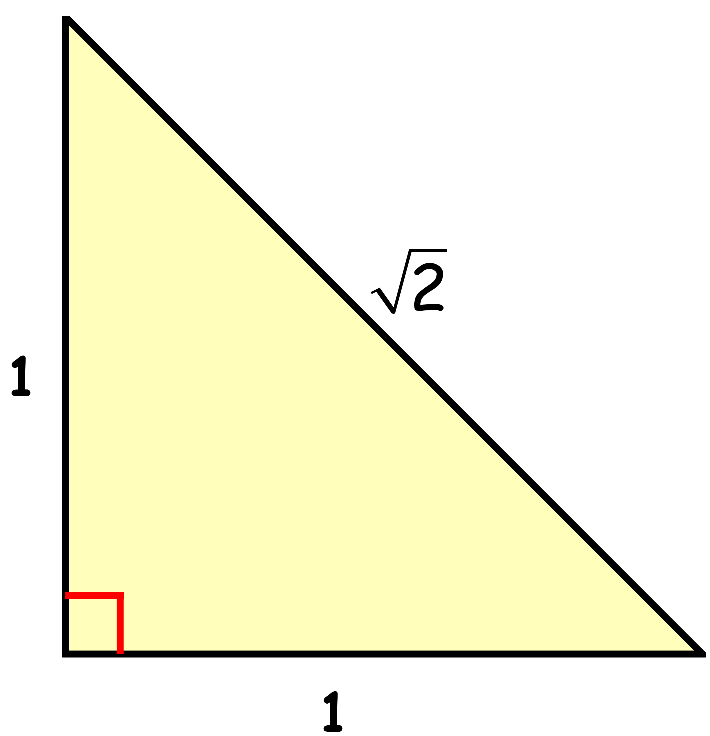
$$d^2 = 2$$

$$d = \sqrt{2}$$




We now have the unit lengths of an isosceles right triangle: $1 - 1 - \sqrt{2}$

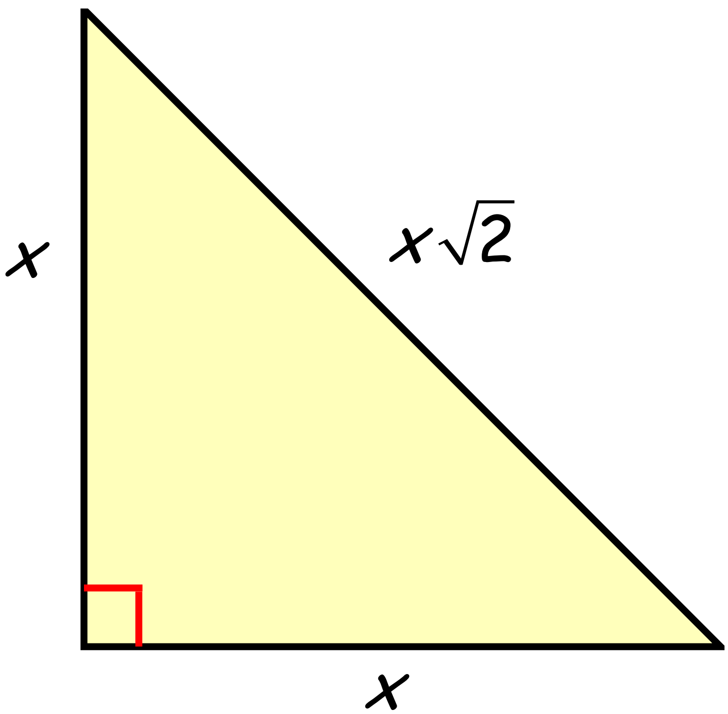
Now we take notice of the angle measures. Since the diagonal of a square bisects the angles (which were originally $90^\circ$), each of the angles in the triangle is $45^\circ$.




Using the idea of proportion and similar triangles, we can multiply all three sides by any non-negative number.

# Special Triangle #1

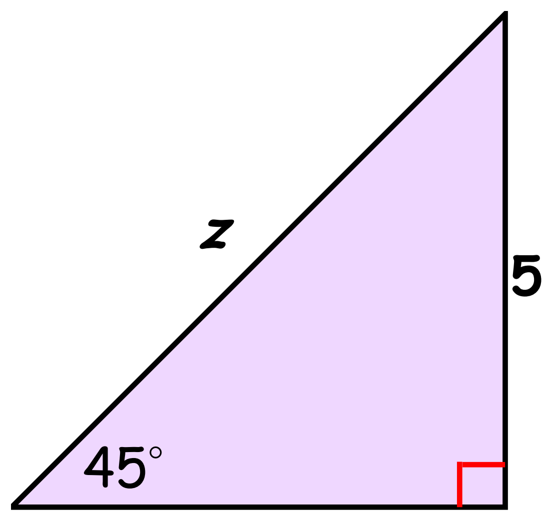
**Theorem:** In a $45^\circ - 45^\circ - 90^\circ$ triangle, the hypotenuse is $\sqrt{2}$ times as long as a leg.




## Example:

Find the value of $z$.

Since this is a 45-45-90 triangle with $x = 5$, the lengths of all 3 sides are $5 - 5 - 5\sqrt{2}$. Since $z$ is the hypotenuse, the longest side, $z = 5\sqrt{2}$

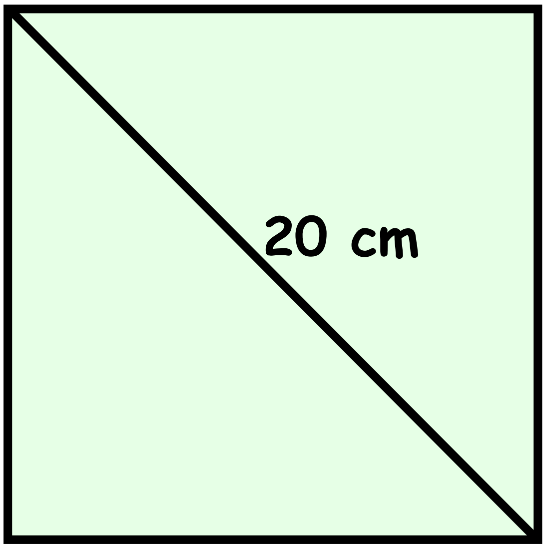



$45^\circ - 45^\circ - 90^\circ$ Rule of Thumb #1:

When going from a leg to a hypotenuse, we multiply by $\sqrt{2}$.

## Example:

The length of a diagonal of a square is 20 centimeters. Find the length of a side of a square.



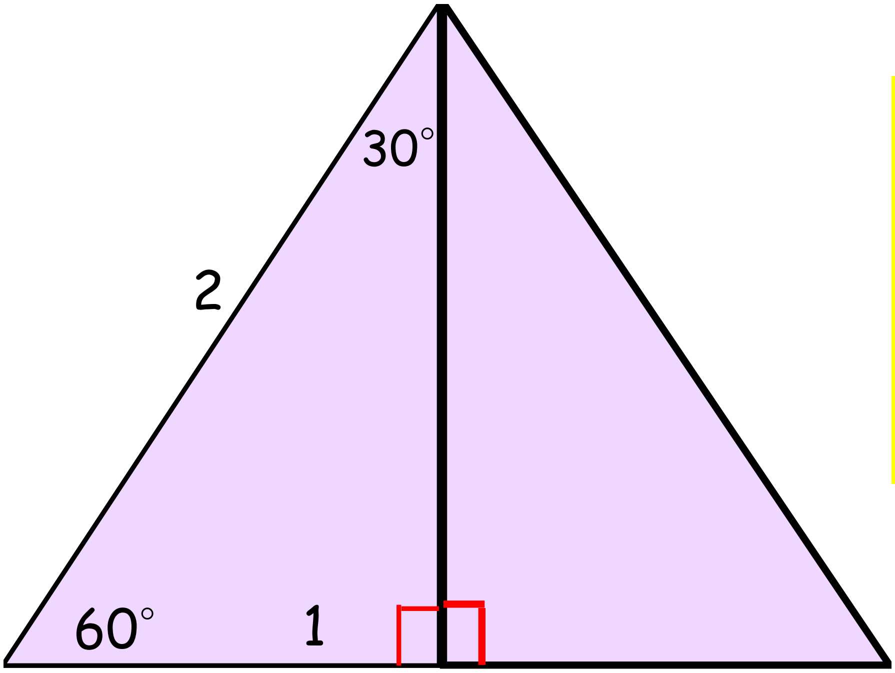

The diagonal of a square forms a 45-45-90 triangle, with the hypotenuse $\sqrt{2}$ times larger than each leg. When going from a hypotenuse to a leg, we DIVIDE by $\sqrt{2}$. So each leg (which make of the length of a side of the square) is $\frac{20}{\sqrt{2}}$. We typically do not prefer radicals in the denominator, so we RATIONALIZE it by multiplying by a clever form of one, namely the radical over itself, then simplify.

$$\frac{20}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)=\frac{20\sqrt{2}}{2}=10\sqrt{2}cm\approx 14.142cm$$

$45^\circ - 45^\circ - 90^\circ$ Rule of Thumb #2:
When going from a hypotenuse to a leg, we divide by $\sqrt{2}$.

Now let's take a look at our second special right triangle.

This one is derived from an equilateral triangle with a side length of 2 units.

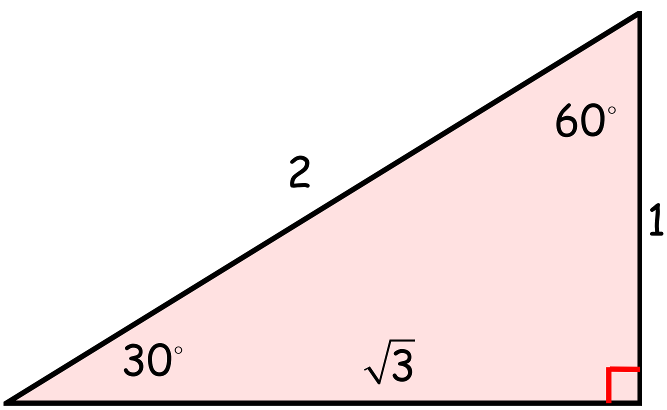



Remember the altitude, *a*, in this case is a perpendicular bisector AND a median. We can find its measure by the Pythagorean Theorem:

$$a^2 + 1^2 = 2^2$$
$$a^2 = 4 - 1$$
$$a = \sqrt{3}$$

## Special Triangle #2

**<u>Theorem:</u>** In a $30^\circ - 60^\circ - 90^\circ$ triangle, the corresponding side lengths are $1 - \sqrt{3} - 2$ .



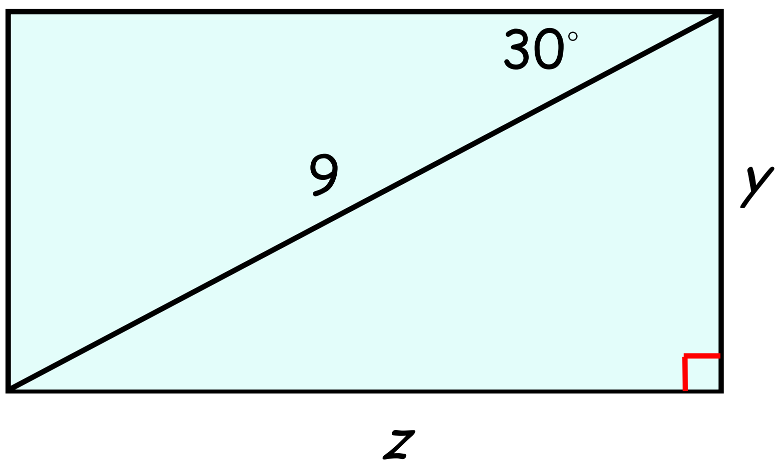

We can also multiply all 3 sides by any non-zero number, or use $x$ as a variable, so the sides become $x - x\sqrt{3} - 2x$

*Remember that the relative sizes of the angles and side lengths correspond.*

For example, the smallest angle is $30^\circ$, so it is opposite the shortest side, the unit length of 1.

# Example:

## Find the value of $y$ and $z$.




Alternate interior angles are congruent, so we get a 30-60-90 triangle on the bottom, with $y$ across from the smallest angle, 30, and 9 across from the largest angle, 90.

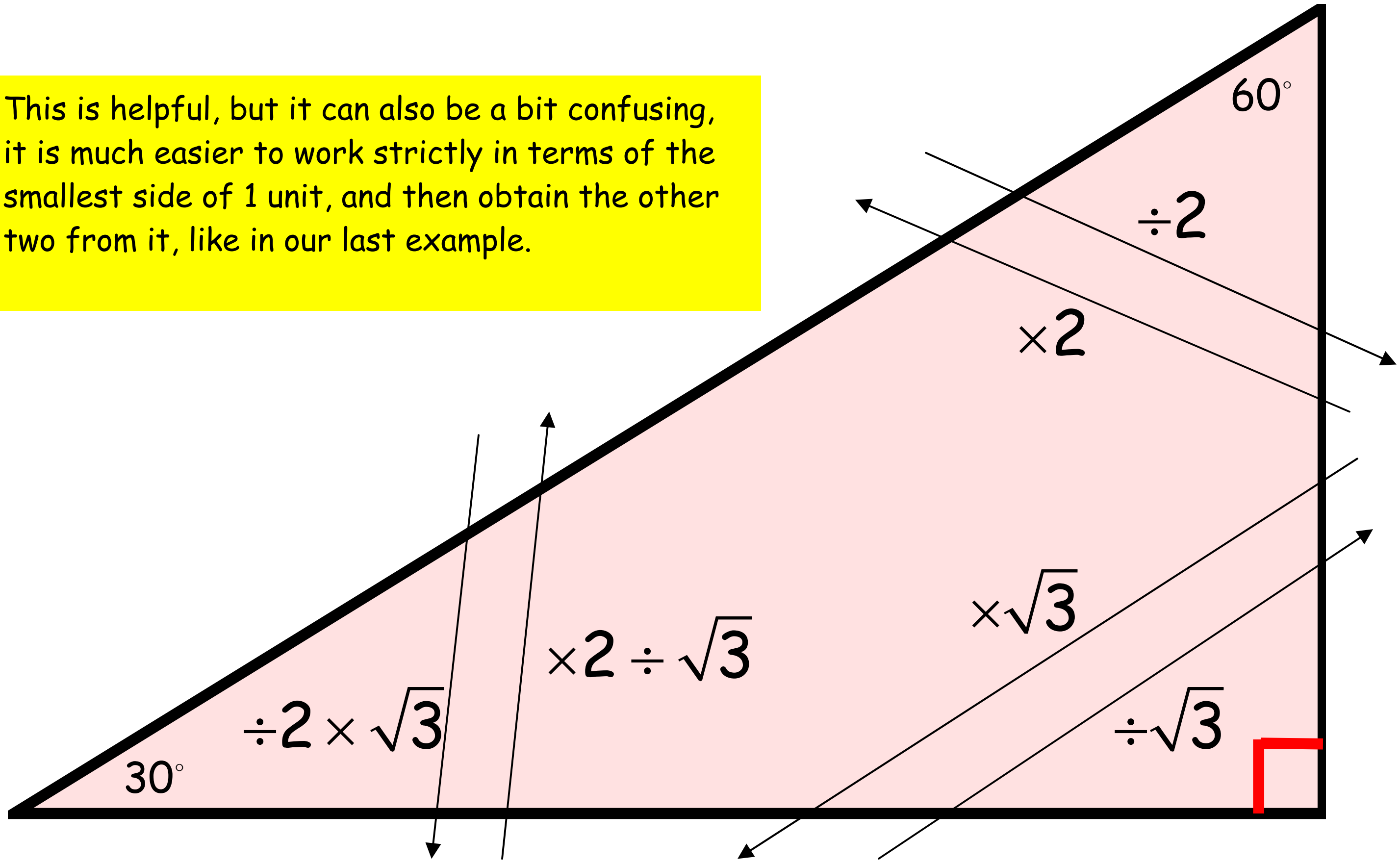
Because the hypotenuse is twice the size of $y$, $y$ must be half the size of the hypotenuse, so $y = \frac{9}{2} = 4.5$

Because the longer leg is always $\sqrt{3}$ times the shorter leg, it is $\left(\frac{9}{2}\right)\sqrt{3} = \frac{9\sqrt{3}}{2} \approx 7.794$

$30^\circ - 60^\circ - 90^\circ$ Rule of Thumb:
Memorize the unit lengths, then appropriately multiply or divide by the correct value to get one side from another!!

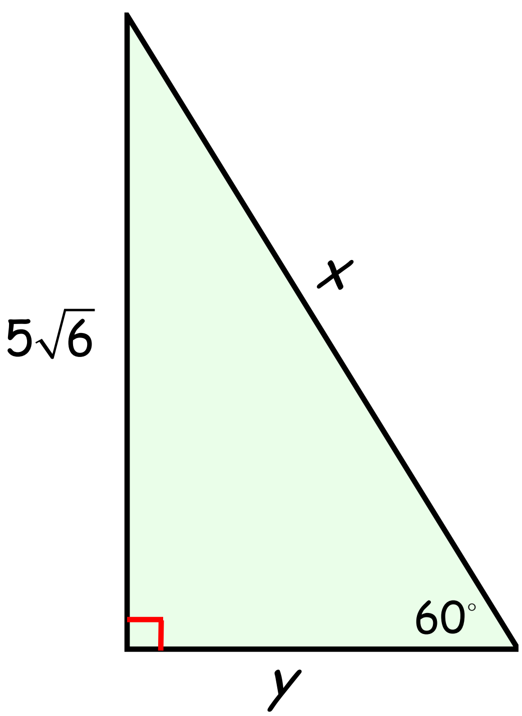
Let me show you what I mean . . .

This is helpful, but it can also be a bit confusing, it is much easier to work strictly in terms of the smallest side of 1 unit, and then obtain the other two from it, like in our last example.




# Example:

Find the value of *x*.




We are going from the middle-length side to the hypotenuse, so, from above, we can multiply by 2 and divide by $\sqrt{3}$.

$$x = 5\sqrt{6}(2)/\sqrt{3} = \frac{10\sqrt{6}}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{10\sqrt{18}}{3} = 10\sqrt{2}$$

OR

We can find the unit side, *y*, first by dividing by $\sqrt{3}$:

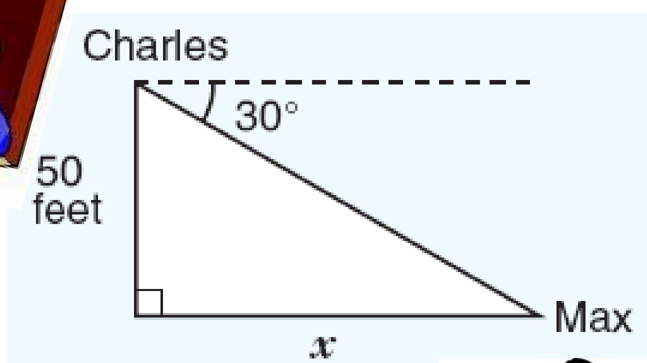
$$\frac{5\sqrt{6}}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{5\sqrt{18}}{3} = \frac{15\sqrt{2}}{3} = y$$

Now we can go from this unit length to the hypotenuse by multiplying by 2:

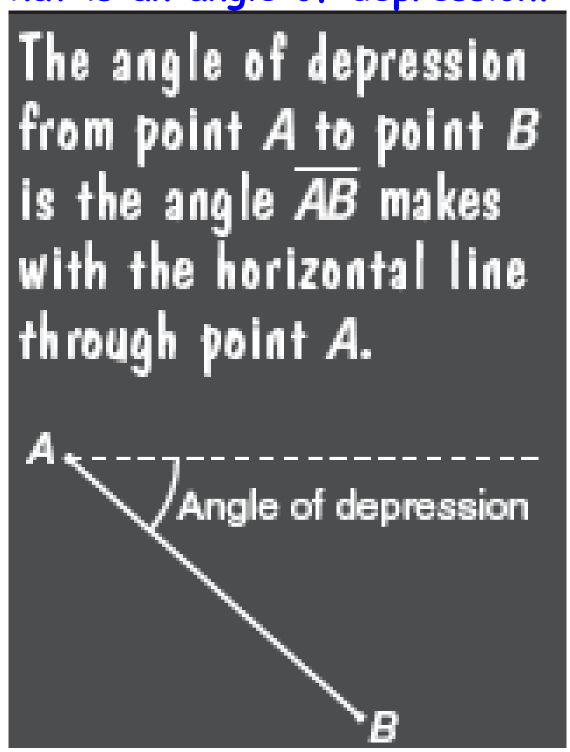
$$x = 2y = (2)\left(\frac{15\sqrt{2}}{3}\right) = \frac{30\sqrt{2}}{3} = 10\sqrt{2}$$

# Say What?!?!

Charles is looking out a window from a point 50 feet above the ground. When Charles looks down at an angle of depression of 30°, he sees his dog Max. To the nearest foot, how far is Max from the base of the building?




**What is an angle of depression?**

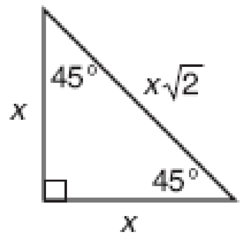
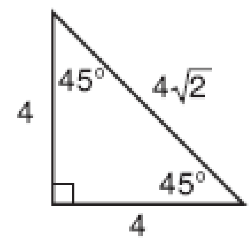
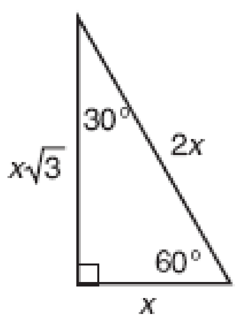
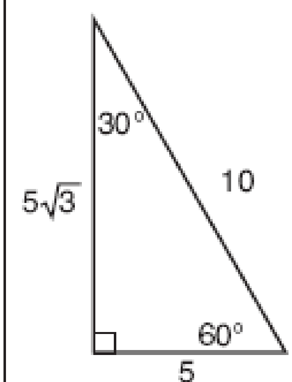



The angle of depression is congruent to the angle of elevation (alternate interior angles). We now have a 30-60-90 triangle with 50 being the shortest side. To get the length of the other leg, which is the distance Max is from the building, we multiply by $\sqrt{3}$.

$x = 50\sqrt{3} \approx 86.6ft \approx 87ft$

So Max is about 87 feet away. “Come here, Max!! Good Boy!”

# Summary

| Special Right Triangle | Side Relationships | Example |
|---|---|---|
| 45°-45°-90° Triangle (Isosceles Right)   | The lengths of the two legs are equal. The length of the hypotenuse is the length of a leg times $\sqrt{2}$. |   |
| 30°-60°-90° Triangle   | The length of the hypotenuse is twice the length of the shorter leg. The length of the longer leg is equal to the length of the shorter leg times $\sqrt{3}$. |   |