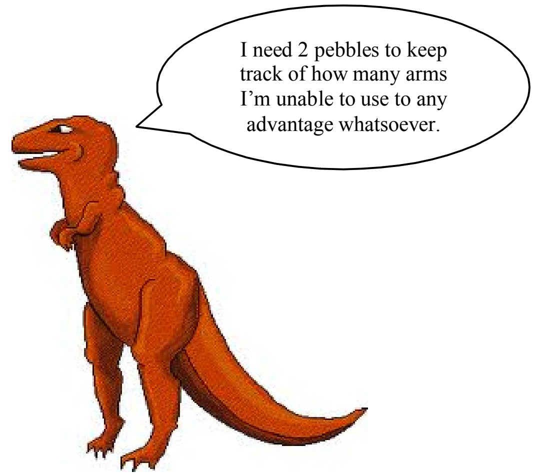
# Chapter 8.1: The Derivative

What is Calculus? It is NOT a 4-letter word! No, it is TWICE that bad, errr, long.




The words "Calculus," "Calcium," "Calculate," and "California," ("California" excluded) all have the same Latin root which means "pebble."

Dinosaurs, if not Cavemen, originally kept track of their things using pebbles. It makes total sense, then, to name the capstone course in High School Mathematics (capitalized for emphasis) after the most primitive, if not dubious, mathematical experiences.

If you had to give the definition of Calculus to someone who was demanding a curt, concise, definitive answer at the expense of taking away your birthdays, I hope you would say that Calculus is

**THE STUDY OF CHANGE**! (you'd better shout it.)

What in our world changes? Pretty much anything and everything (except for boring things, like square-headed Germans and Chameleons' skin color . . . wait, I take that second one back.)

Calculus is not only valuable in studying the difference in political candidates' views pre- and post-election, but we can also use calculus to optimize situations: like how to get the BEST grade with the LEAST amount of effort expended.

We will begin our study of calculus by looking at the simple idea of SLOPES!! Be careful not to lose your footing. Whenever we discuss a rate, that is, how one quantity changes with respect to another quantity, we're talking about a slope, which is nothing more than a rate of change. For instance, when you talk about miles per hour, feet per second, or glances per piercings, you're talking about a slope, and consequently, CALCULUS!

Mathematically, the simplest object about which to discuss slopes are straight lines.

**Example 1:**
Find the slope at **any** and **every** point of the line passing through the points $(-2,3)$ and $(5,6)$.

**Example 2:**
Find the slope of the line given by the general form equation $5x-3y=2$ at $x=3$, then find the line's instantaneous rate of change at $x=-50$, then find the line's average rate of change on the interval $x \in [-50,3]$. What would happen to the slopes at a particular *x*-value if the given line was vertically shifted up or down?

**Slope of a Line**

The slope of a line of the form $f(x)=mx+b$ has a constant slope of *m* at **an**y and **all** points on the line. This means at any point on the line, an increase of one *x*-unit will correspond with a change of *m y*-units. At **any** and **all** points on a line, the **average rate of change** is equal to the **instantaneous rate of change**.

The slope of a horizontal line of the form $f(x)=b$, where *b* is any constant, is equal to zero.
The slope of a vertical line of the form $x=a$, where *a* is any constant, is either $\infty$ or $-\infty$.

Finding slopes of lines is easy, but what if we wanted to find the slope of a curve?

**Example 3:**
Given $f(x)=x^2+1$, what does it mean to talk about the slope of this parabola? Where is the slope of the graph negative? positive? zero? What can we say about the graph in each case? Given $f(x)=x^2+1$, what is the difficulty in finding the slope of the graph at $x=-3$?

Slopes of lines are easy to calculate using a **difference quotient**, but you need TWO points on the line, and we only have one.

If we calculate the slope of the line between two points on a curve, we get the **average rate of change** of that curve on that interval. This line is called a **secant line**.

The slope that we really want in **Example 3** is the slope of the **tangent line** at $x=-3$, that is the line that just "kisses" the graph at the point $(-3,10)$ from underneath the parabola. **In general, the slope of the tangent line at a point gives us the slope of the curve at that point and is the instantaneous rate of change of the curve at that point.**

| |
|---|
| Interesting Fact: Calculus was discovered independently in two different parts of the world at roughly the same time. In 1674, Leibniz was in Germany trying to figure out how to find the slope of a tangent line to a curve at a single point, while in 1666 in England, Isaac Newton was trying to determine the instantaneous rate of change of a particle in motion. It turns out that the solution to both of their questions was the same—Calculus! |

**Important Ideas**

| | |
|---|---|
| Average Rate of Change<br>=<br>Slope of a Secant Line | Instantaneous Rate of Change<br>=<br>Slope of a Tangent Line |

Here's a hint of how our mathematical forefathers got around finding the slope of the tangent line with only one point.

**Example 4:**
Given $f(x)=x^2+1$, find the slope of the secant line between

(a) $x=-3$ and $x=5$ (b) $x=-3$ and $x=2$ (c) $x=-3$ and $x=0$ (d) $x=-3$ and $x=-2$

Can you see what's happening to the secant lines above as our second $x$-value converges (approaches) $x=-3$? It appears that the slope of the secant line is approaching the slope of the tangent line!! In theory, we can get a good approximation of the slope of the tangent line at $x=-3$ by choosing another point on the graph of $f(x)=x^2+1$ really, really, really close to $x=-3$ . . . or, we can be more clever about it using a limiting process.

**Example 5:**
Given $f(x) = x^2 + 1$, find the slope of the **secant line** between $(-3,10)$ and $(-3+h, f(-3+h))$, where $h$ is some small $\Delta x$. After simplifying the **difference quotient**, analyze what happens to expression as $h$ approaches zero.

**Example 6:**
Using the same method from **Example 5**, find the instantaneous rate of change of $f(x) = x^2 + 1$ at $x = 2$.

This limit method works pretty well, but it does require setting up the difference quotient each time we want to find the slope at a different point, but not a bad start. What if we want to find the slope of a ***different*** function?? Well, in general can do this for **ANY** point $(x, f(x))$ on the graph of any function $y = f(x)$.

**VERY Important Idea**

The slope of the tangent line to a graph at $(x, f(x))$ is the limit of the slope of the secant line between $(x, f(x))$ and $(x+h, f(x+h))$ as $h$ goes to zero.

We can state this mathematically as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Where $f'(x)$ is called the "slope function" or the **derivative** function of *f*, and is read as "*f* prime of *x*." Calculating the derivative function is called **differentiation**.

- The derivative is a function that gives the slope of the tangent line of another function at a particular point. It also provides the instantaneous rate of change of this function at this point.
- We use other equivalent notations for the derivative function: $f'(x)$, $y'$, $\frac{dy}{dx}$ (Leibniz notation)

**Example 7:**
Given $f(x) = x^2 + 1$, find the derivative function $f'(x)$ using the limit definition above, then evaluate and interpret the following: $f'(-3)$, $f'(2)$, and $f'(0)$. At what point(s) on the graph of $f(x) = x^2 + 1$ will the slope be $\frac{13}{4}$?

**Example 8:**
What happens to the derivative function and/or the specific slopes calculated above if the original function $f(x) = x^2 + 1$ was vertically shifted? Why? Is this result true for ALL functions affected by such a vertical shift??

**Another Important Idea:**

Two functions $f(x)$ and $g(x)$ differ only by a constant, that is $f(x)=g(x)+C$ if and only if $f'(x)=g'(x)$ for all *x* in the domain of *f* and *g*.

As much fun as finding the derivative using the formal limit definition, the "good" news is that we don't have to go through that long, laborious, lugubrious process each and every time. In fact, if we don it one time using a general form, we get a pattern called the **Power Rule** for differentiation.

**Example 9:**
Complete the chart below by recognizing the pattern:

| | | | |
|---|---|---|---|
| $y=3$ | $y'=0$ | $y=0$ | $y'=0$ |
| $y=x$ | $y'=1$ | $y=2x-5$ | $y'=2$ |
| $y=x^2$ | $y'=2x$ | $y=6x^2+5x-7$ | $y'=12x+5$ |
| $y=x^3$ | $y'=3x^2$ | $y=4x^3-7x^2-8x+\pi$ | $y'=$ |
| $y=x^4$ | $y'=$ | $y=-3x^4-7x^2+9x$ | $y'=$ |
| $y=x^5$ | $y'=$ | $y=6x^5-4x^4+3x-1$ | $y'=$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $y=x^n$ | $y'=$ | $y=ax^n+C$ | $y'=$ |

**The Power Rule**

If a function is of the form $y=ax^n$, where $a\neq 0$ and $n\in\mathbb{R}$, then $y'=an\cdot x^{n-1}$

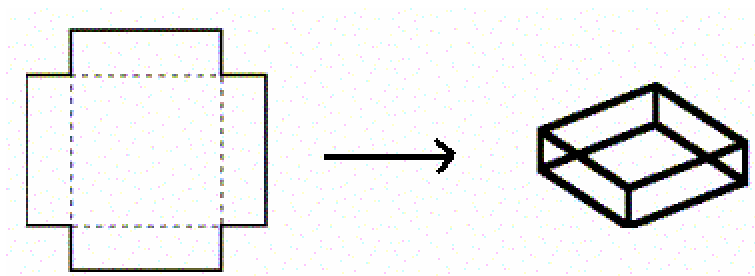
**Example 10:**
For $f(x)=\frac{2}{3}x^3+2x^2-30x-\sqrt{3}$, find $f'(x)$, $f'(-2)$, $f'(1)$, the find the *x*-coordinate(s) of any horizontal tangent lines.

In Example 10, we saw that the coefficients need not be integers. Additionally, when using the power rule to find derivative functions, the exponents don't even need to be Whole Numbers. As long as each term in the function can be written or rewritten in the form $ax^n$, the power rule works.

**Example 11:**
If $f(x) = 2\sqrt{x} - \frac{3}{x^2} + \frac{5}{\sqrt[3]{x}} - \frac{2x+2}{x}$, find $f'(x)$ by eventually using the power rule. Write your final answer without negative or rational exponents.

We can use the power of the derivative to find optimal quantities.

**Example 12:**
Find the volume and dimensions of the largest open-topped box that can be cut from a 12-in by 12-inch piece of cardboard.



There are some other useful derivatives to know for which the power rule does not help us find.

**Example 13:**

Sketch two cycles of the graph of $y = \sin x$, then on the same set of axes, sketch the graph of $y' = \frac{dy}{dx}$ by analyzing the slopes of $y = \sin x$ at the critical values. Do you recognize the equation of the derivative function??

**Other Useful Derivatives**

| $y$ | $y'$ |
|---|---|
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |
| $e^x$ | $e^x$ |
| $\ln x$ | $\frac{1}{x}$ |

We can expand the power of the power rule to deal with products or quotients of variable functions.

**Product and Quotient Rules**

| **Product Rule** | **Quotient Rule** |
|---|---|
| If $y = f(x) \cdot g(x)$, then $y' = f'(x)g(x) + f(x)g'(x)$ | If $y = \frac{f(x)}{g(x)}$, $g(x) \neq 0$, then $y' = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$ |

**Example 14:**

Find the derivative of each of the following functions:

(a) $y = 3x^2 \ln x$

(b) $f\left(x\right) = \tan x$

(c) $g\left(x\right) = 5x^2 e^x$

(d) $y = \dfrac{\left(3x+1\right)^2}{2x+3}$

That last one had an interesting numerator, a **composite function**. What if it had had an exponent of, let's say, 12 instead of 2, would we want (or be able to) expand it first??!! No! If only there was another way. There is . . . the **Chain Rule**.

**The Chain Rule** (for composite functions)

If $y = f\left(g\left(x\right)\right)$, then $y' = f'\left(g\left(x\right)\right) \cdot g'\left(x\right)$

*For the Chain Rule, rewriting is sometimes the key so that you can see the "layers" embedded in the function. In general, the number of embedded layers will equal the number of factor "links" in the chain.

**Example 15:**

Find the derivative of each of the following using the chain rule:

(a) $y=(3x+1)^2$

(b) $f(x)=(3x+1)^{12}$

(c) $g(x)=\sqrt[3]{(3x+1)^2}$

(d) $y=\sin^2(3x)$

(e) $y=\ln e^{x^2}$

(f) $y=\sin\left(e^{\cos 2x}\right)$

(g) $y=\ln(\ln 4x)$

(h) $h(x)=x^2\sin 5x$

(i) $q(x)=\dfrac{e^{4x}}{\cos 3x}$