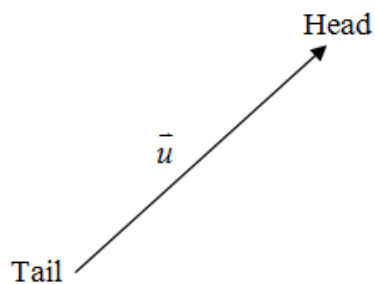


# Chapter 7.4: Vectors

In many mathematical applications, quantities are determined entirely by their magnitude. When calculating the perimeter of a rectangular field, determining the weight of a box, or recording the outdoor temperature, we are only interested in a single number. These quantities are called **scalar quantities**.

Other mathematical quantities, however, require two pieces of information to describe them, both a **magnitude** AND a **direction**. These are called **directional quantities**. For example, to describe an object's velocity, we need both the speed and the direction of travel. Speed itself is a scalar quantity. One convenient way to represent these directional quantities is with a **vector**.

We can geometrically describe a vector by a directional line segment. Resembling a ray, a geometric vector is a fixed-length, with an initial point, called a **tail**, and a terminal point, called a **head**. We can name a vector by using a bold, lower-case letter, such as  $\mathbf{u}$  or  $\mathbf{v}$ , or, since we do not write in bold, by the notation  $\vec{u}$  or  $\vec{v}$ . A vector can also be name using tail-to-head method, such as  $\overline{AB}$ , with  $A$  being the initial point and  $B$  being the terminal point.



The length of the vector is called the magnitude or norm of the vector and is denoted by  $|\vec{u}|$  or often  $\|\vec{u}\|$ . The direction in which the vector actually points indicates the direction in Euclidean space.

Two vectors are considered equal if they have the same magnitude and direction. This means we can "slide" the vector to the left anywhere around on this page, and as long as they stay parallel and the same length, all will be equal to each other.

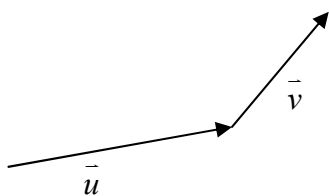


We can think of a vector as a **displacement** of an object in a plane. Equal vectors in different places can be thought of the same displacement applied to different objects at different places in the plane.

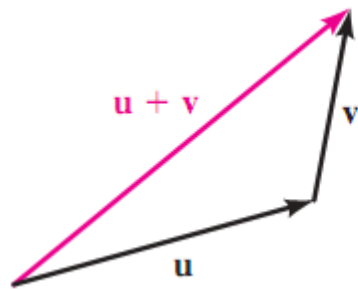
Of course, we can apply these displacements consecutively to the same object.

### Example 1:

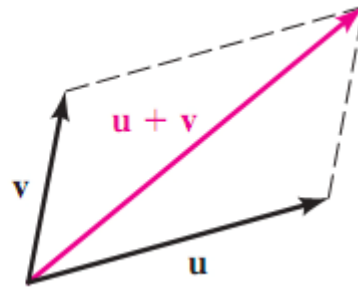
If vector  $\vec{u}$  is first applied to an object followed by  $\vec{v}$ , find the **resulting displacement vector**,  $\vec{r}$ .



The process in the previous example is called vector addition by the **head-to-tail method**. Vectors can also be added by the **parallelogram method**, with the sum being the vector making the diagonal from the vertex to the two vectors being added. Regardless of how we think about it geometrically, we can denote this as  $\vec{r} = \vec{u} + \vec{v}$ .



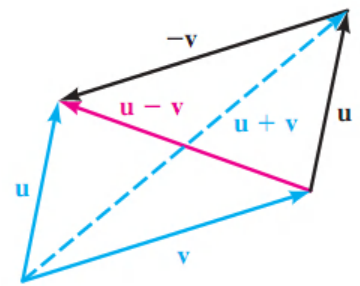
head-to-tail method



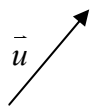
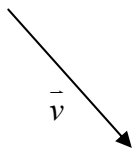
parallelogram method

In addition to adding two vectors, we can also change the length and/or direction of a vector by multiplying that vector by a **scalar multiple**. If  $a$  is any real number and  $\vec{v}$  is a vector, we obtain a new vector  $a\vec{v}$  with magnitude  $|a| \cdot \|\vec{v}\|$ . If  $a > 0$ , the new vector has the same direction as  $\vec{v}$ .

If  $a < 0$ , the new vector points in the opposite direction of  $\vec{v}$ , with the head and tail interchanging. If  $a = 0$ , then  $a\vec{v} = \vec{0}$ , the zero vector. The **difference** of two vectors  $\vec{u}$  and  $\vec{v}$  is defined as  $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$ .

**Example 2:**

Vectors  $\vec{v}$  and  $\vec{u}$  are shown below. Find the following.



(a)  $2\vec{u}$

(b)  $\frac{1}{2}\vec{v}$

(c)  $-\vec{u}$

(d)  $-\frac{1}{4}\vec{v}$

(e)  $\vec{v} - \vec{u}$

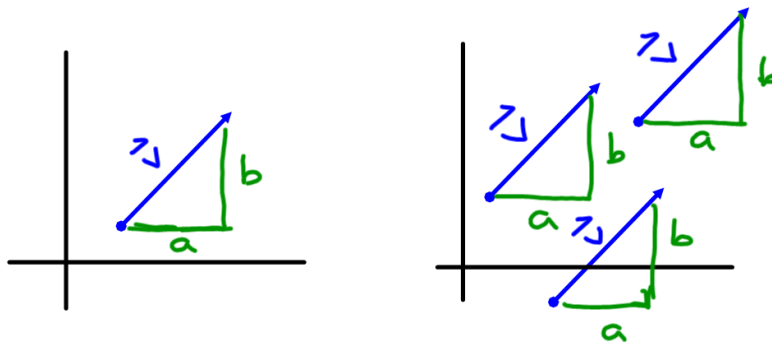
(f)  $3\vec{u} - 2\vec{v}$

All this drawing of vectors all over the page is fun, but a little bit arbitrary. We would like to standardize our process so that it becomes systematic, routine, and easy to analyze. To do this, we will plot our vectors on the coordinate/Cartesian plane. This will allow us to decompose any vector into its horizontal and vertical components. It will also allow us to use all of the wonderful trigonometry we've learned to assist us in our analysis and problem-solving endeavors.

We can describe any vector in a plane analytically by defining its horizontal and vertical component. In general, we can move  $a$  units **horizontally** from the tail of vector  $\vec{v}$ , then move  $b$  units **vertically** to get to the head of vector  $\vec{v}$ . We then represent vector  $\vec{v}$  as an ordered pair set off in chevrons.

$$\vec{v} = \langle a, b \rangle$$

Depending on where the tail of the vector is, this vector has many equivalent forms in a Euclidean plane, as shown below.



**Example 3:**

Find the component form of the vector  $\vec{u}$  whose tail is at  $(-2, 6)$  and whose head is at  $(3, 8)$ .

**Example 4:**

If the vector  $\vec{v} = \langle 3, 8 \rangle$  is sketched with its tail at  $(1, 5)$ , at what point is its head?

**Example 5:**

Sketch different representations of vector  $\vec{r} = \langle 2, -3 \rangle$  with initial points at  $(0, 0)$ ,  $(2, 2)$ ,  $(-2, -1)$ , and  $(-1, 4)$

The vector from the above example whose tail was at the origin is called a vector in **standard position**.

### Equality and Magnitude of Vectors

Two vectors  $\vec{u} = \langle a_1, b_1 \rangle$  and  $\vec{v} = \langle a_2, b_2 \rangle$  are **equal** if and only if they have the same magnitude and direction. This means their components are equal:  $a_1 = a_2$  and  $b_1 = b_2$ .

The **magnitude** (or **norm**) of a vector  $\vec{v} = \langle a, b \rangle$ , using the Pythagorean Theorem, is given by

$$|\vec{v}| = \|\vec{v}\| = \sqrt{a^2 + b^2}$$

### Example 6:

Sketch the following vectors in standard position, then find the norm of each vector.

(a)  $\vec{u} = \langle 3, -5 \rangle$

(b)  $\vec{v} = \langle 0, -4 \rangle$

(c)  $\vec{w} = \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$

We have seen how to geometrically add vectors, subtract vectors, and multiply vectors by scalar multiples. We will now see how we can perform these tasks algebraically.

### Algebraic Operations on Vectors

Given two vectors  $\vec{u} = \langle a_1, b_1 \rangle$  and  $\vec{v} = \langle a_2, b_2 \rangle$ , and  $c \in \mathbb{R}$ ,

$$\vec{u} + \vec{v} = \langle a_1 + a_2, b_1 + b_2 \rangle$$

$$\vec{u} - \vec{v} = \langle a_1 - a_2, b_1 - b_2 \rangle$$

$$c\vec{u} = \langle ca_1, cb_1 \rangle$$

### Example 7:

If  $\vec{u} = \langle 3, -2 \rangle$  and  $\vec{v} = \langle -2, 5 \rangle$ , find

(a)  $\vec{u} + \vec{v}$

(b)  $\vec{u} - \vec{v}$

(c)  $3\vec{u} - 2\vec{v}$

(d)  $-\frac{2}{3}\vec{u} + \frac{4}{5}\vec{v}$

## Zero and Unit Vectors

The **zero vector**  $\vec{0} = \langle 0, 0 \rangle$  is the identity vector for vector addition, such that  $\vec{u} + \vec{0} = \vec{u} = \vec{0} + \vec{u}$

$\vec{u}$  is called a **unit vector** if  $|\vec{u}| = 1$ , that is, its magnitude is one unit. In general, if  $\vec{v}$  is not the zero vector, then

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{|\vec{v}|} \vec{v}$$

is a **unit vector in the direction of  $\vec{v}$** .

The vector  $\vec{w} = \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$  from **Example 6c** is one example of a unit vector.

### Example 8:

Find a unit vector,  $\vec{u}$ , in the direction of  $\vec{v} = \langle -2, 3 \rangle$ , then verify it has a length of 1. Draw it in standard position.

Two useful unit vectors are  $\vec{i} = \langle 1, 0 \rangle$  and  $\vec{j} = \langle 0, 1 \rangle$ , which are called the **unit component vectors**.

This gives us an alternative way to define vectors.

### Vectors in terms of $\vec{i}$ and $\vec{j}$

A vector  $\vec{u} = \langle a, b \rangle$  can be equivalently be expressed as a **linear combination** in terms of  $\vec{i}$  and  $\vec{j}$  by the vector

$$\vec{u} = \langle a, b \rangle = a\vec{i} + b\vec{j}$$

### Example 9:

If  $\vec{u} = \langle -5, 6 \rangle$  and  $\vec{v} = 3\vec{i} - 2\vec{j}$

(a) Write vector  $\vec{u} = \langle -5, 6 \rangle$  in terms of  $\vec{i}$  and  $\vec{j}$ .

(b)  $4\vec{u} - 3\vec{v}$  in terms of  $\vec{i}$  and  $\vec{j}$

If we have a vector  $\vec{v}$  in standard position, then the **direction of the vector** can be expressed as the positive angle,  $\theta$ , between the positive  $x$ -axis and the vector. If we then know the magnitude of the vector, we can decompose the vector into its horizontal and vertical components.

### Horizontal and Vertical Components of a Vector

If  $\vec{v}$  is a vector with magnitude  $|\vec{v}|$  and direction  $\theta$ , then  $\vec{v} = \langle a, b \rangle = a\vec{i} + b\vec{j}$ , with

$$a = |\vec{v}| \cos \theta \quad \text{and} \quad b = |\vec{v}| \sin \theta$$

The vector then becomes  $\vec{v} = \langle |\vec{v}| \cos \theta, |\vec{v}| \sin \theta \rangle = |\vec{v}| \cos \theta \vec{i} + |\vec{v}| \sin \theta \vec{j}$

From the above formula, it follows that the unit vector in the direction of  $\vec{v}$  is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \langle \cos \theta, \sin \theta \rangle \quad (\text{look familiar??})$$

#### Example 10:

A vector  $\vec{k}$  has a length and direction of  $\frac{5\pi}{6}$ . Find the horizontal and vertical components, and write  $\vec{k}$  in chevron and  $\vec{i}, \vec{j}$  notation.

#### Example 11:

Find the direction and the magnitude of the following vectors (use your calculator on (b) & (c))

(a)  $\vec{r} = -\vec{i} + \sqrt{3}\vec{j}$

(b)  $\vec{u} = \langle 2, -3 \rangle$

(c)  $\vec{v} = -5\vec{i} - 2\vec{j}$

We can use our new representations and algebra of vectors to model and easily answer questions involving velocity and force.

**Example 12:**

(Calculator) An airplane heads due north at 300 mph. It experiences a 40 mph crosswind flowing in the direction  $30^\circ$  East of North.

- (a) Express the velocity  $\vec{p}$  of the airplane relative to the air, and the velocity  $\vec{w}$  of the wind, in component form.
- (b) Find the true velocity of the plane as a vector,  $\vec{r}$ .
- (c) Find the true speed and direction of the plane.

**Example 13:**

Corny the Unicorn wants to launch his boat from one shore of a straight river and wants to land at the point directly across on the opposite shore. If the speed of the boat (relative to the water) is 15 mph, and the river is flowing east at a rate of 6 mph, in what direction should Corny point his boat upstream to arrive at his desired landing point?

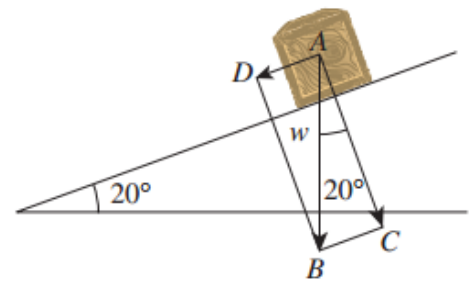
A **force** can be simply described as a push or pull of an object in a certain direction. We can measure a force in pounds (lbs) or newtons (N). We can represent a force by a vector. For example, a 220 lb math teacher exerts a force of 220 lbs downward on the ground. If more than one force acts on an object at the same time, the **resultant force** on the object is the vector sum of those forces.

**Example 14:**

Two forces  $\vec{G}$  and  $\vec{E}$  are acting on an object. Force  $\vec{G}$  has a force of 20 lb in a direction of  $150^\circ$ . Force  $\vec{E}$  has a force of 10 lb in a direction of  $45^\circ$ . Find the resultant force acting on the object.

**Example 15:**

A force of 30 pounds is juuuuust enough to keep the box at right from sliding down the ramp inclined at  $20^\circ$ . If Corny the Unicorn is hiding in the box along with a bunch of rocks (the box and rocks weigh a combined 86 pounds), find Corny's weight!



The force of gravity  $\vec{AB}$  has a component  $\vec{AC}$  that holds the box against the surface of the ramp and another component  $\vec{AD} = \vec{CB}$  that wants to push the box down the ramp.