Chapter 7.3: Plane Curves and Parametric Equations

Imagine hitting a golf ball and watching its flight path until it lands. We can write rectangular equations that model the height of the ball as a function of the distance travelled, but often we are interested in analyzing each of these separately as a function of time. To do this, we would need two separate equations, one to model the height of the ball (call it *y*) as a function of time, *t*, and another to model the distance the ball travels (call it *x*) as a function of time, *t*.

This is the idea behind parametric equations: graphing y against x, enabeling us to actually see the path of the ball, but while maintaining the aspect of time. In general, parametric equations are a pair of equations that involve a third, independent variable, though it doesn't always have to be time.

Plane Curves and Parametric Equations

Let $x = f(t)$ and $y = g(t)$, where *f* and *g* are two functions whose common domain is some interval *I*. The collection of points $(x, y) = (f(t), g(t))$ is called a **plane curve**. The equations $x = f(t)$ and $y = g(t)$, where *t* is in *I*, are called **parametric equations** of the curve, and the variable *t* is called the **parameter**.

We can use parametric equations to write component equations for projectile motion given some intial condidtions. Specifically, for a projectile launched from a point (h, k) at an angle of θ with the horizontal and an initial velocity v ft/sec, its position at time t seconds (neglecting air resistance and hitting flying birds) is given by:

$$\boxed{\begin{aligned} x(t) &= (v\cos\theta)t + h \\ y(t) &= -16t^2 + (v\sin\theta)t + k \end{aligned}}$$

Example 1:
During a golf game, Corny the Unicorn estimates the distance to the pin to be 600 feet. His swing provides an initial velocity of 160 ft/s to the ball at an angle of 28 degrees above the horizontal.
(a) Write the parametric equations describing the flight path of the ball in the air, then graph and analyze them on your calculator. (b) What is the maximum height of the golf ball? (c) How far horizontally is the ball from where he hit it when it reaches its maximum height? (d) Will he hit the pin in the air?

Example 2:
A particle moves through the *xy*-plane. Its position at given in feet at time *t* seconds is modeled by the equations below. Without a calculator, make a table, and sketch the curve the particle follows, indicating its direction. Then eliminate the parameter. Verify on your calculator.

$$x = t^2 - 4 \text{ and } y = \frac{t}{2},\ -2 \le t \le 3$$

Example 3:
Using a calculator in parametric mode, (a) what do you notice about the graphs of $x = 4t^2 - 4$ and $y = t$, $-1 \le t \le 1.5$ (b) What do you notice about the graphs of $x = 4(2\sin t + 1)^2 - 4$ and $y = 2\sin t + 1$, $-1.571 \le t \le 0.253$

While rectangular equations on restricted intervals show the _____________, parametric equations show

the __________________________, _______________________, and ____________________________.

Example 4:
For the following parametric equations, (a) determine the domain, then find (b) $\lim_{t\to\infty} x(t)$ and (c) $\lim_{t\to\infty} y(t)$, then (d) sketch the curve using your calculator with appropriate *t*-values, then (e) eliminate the parameter and verify the path.

$$x = \frac{1}{\sqrt{t+1}},\ y = \frac{t}{t+1}$$

Example 5:
Using your calculator (a) sketch the parametric curve (radian mode), then (b) using your algebra II knowledge of conic sections, write the rectangular equation of the curve in standard form, then (c) eliminate the parameter using Pappa PID.

$$x = 2 + 3\cos t\ ,\ \ y = -1 + 2\sin t$$

Example 6:
(a) Write the equation of the line in slope-intercept form whose slope is -3 that passes through the point $(-2,4)$, then (b) sketch it on the restricted interval $-2 \le x \le 2$. (c) For each of the following, write three different sets of parametric equations and parameter restrictions that give us the path described above and describe what is the same and different about each.

I. $x = t$ II. $x = -2t$ III. $x = e^t - 3$

For complicated curves, graphing calculators, in correct viewing windows, can be useful for graphing them.

Example 7:
Using your calculator, describe the motion of a particle whose position in the in the *xy*-plane is modeled by the equations $x(t) = t + \sin 2t$ and $y(t) = t + \cos 5t$, $t \in [-6,6]$,.

We have now seen how both polar equations and parametric equations model complicated curves, especially curves that fail the vertical line test, much more easily. We've also seen how we can model rectangular equations in parametric form. Believe it or not, we can also model polar equations in parametric form (although we lose the "speed" aspect).

Example 8:
Consider the polar equation $r = 2 + 2\sin\theta$, $0 \le \theta \le 2\pi$. Express the equation in parametric form, then graph the equation in parametric form.