Chapter 6.6: The Super Pythagorean Theorem Or The Law of Cosines (& Area)

You're in for a special treat this section.

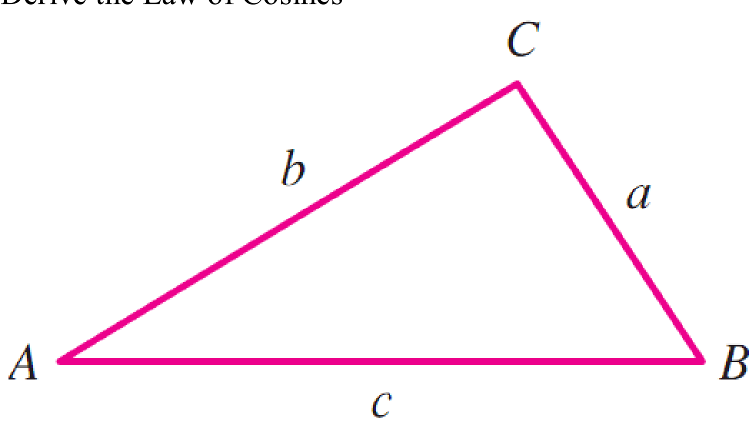
We all know how important, useful, and utterly amazing the Pythagorean Theorem is. Unfortunately, it only works for right triangles.

What if there was a Pythagorean-esque Theorem that worked for ALL triangles? Well, guess what! There is! And it's called the **Law of Cosines**.

Just like the Pythagorean Theorem works well when dealing with side lengths, so does the Law of Cosines, but the LOC also works well when we only have two sides and an included angle.

We will be using the Law of Cosines to solve triangles given **SSS** or **SAS** (although it can also be used in the SSA ambiguous case).

Example 1:
Derive the Law of Cosines



The Law of Cosines

For ANY triangle *ABC*,

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Example 2:

Solve triangle *ABC* if $a = 11$, $b = 5$, $C = 20^\circ$ using the Law of Cosines

Be very, very, very careful if you are using the Law of Sines to find Angles (this should ONLY be done in the SSA case, after you've determined the number of solutions). Because and angle and its supplement have the same sine value, you might not always get the correct angle.

Example 3:

Solve triangle *ABC* if $a = 11$, $b = 5$, $C = 20^\circ$ using the Law of Cosines to find *c*, then (a) use the Law of SINES to find angle *A*, then (b) use the Law of Sines to find angle *B*

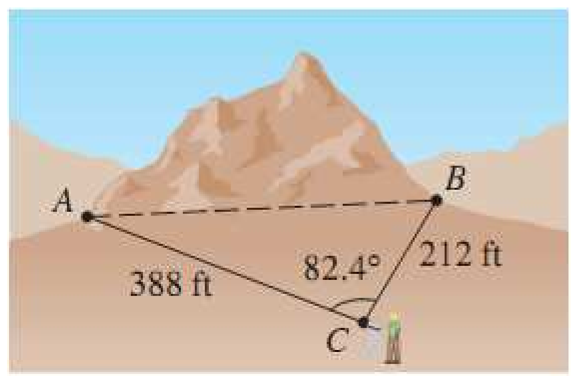
**The Law of Sines CAN be used to find angles in non-ambiguous cases, but care should be taken to find the SMALLER of the two remaining angles. If you're not committed to checking and are desiring a correct answer, use the Law of Cosines.

Example 4:

Solve triangle ABC if $a = 9$, $b = 7$, and $c = 5$ using any methods.

Example 5:

A tunnel is to be built through a mountain. To determine the approximate length of the tunnel, an ambitious math student with a homemade transit takes to measurements from a fixed point (as shown in the figure). Use the student's data to determine the length of the tunnel.



Example 6:

An airplane pilot sets out from the NB Municipal airport on a heading of 20°, flying at 200 miles per hour. After an hour, he changes his heading to 40°. After another half hour on this new heading, he lands at his house. What is his distance from the airport to home. In the morning, when he leaves the house in his plane to return to the airport, on what heading should he fly to get there directly?

One last thing regarding triangles that we need to know how to find: Area

Formulas for Areas of Triangles

Right Triangles $\text{Area} = \frac{1}{2}bh$	**SAS** $\text{Area} = \frac{1}{2}ab\sin C$	**SSS** $\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$
Equilateral Triangles $\text{Area} = \frac{\sqrt{3}}{4}s^2$	**ASA or AAS** $\text{Area} = \frac{1}{2}a^2\frac{\sin B\sin C}{\sin A}$	

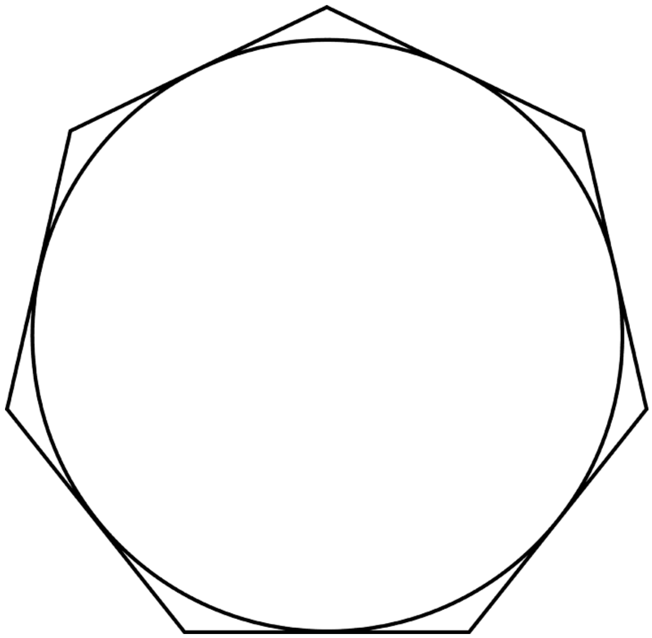
Example 7:

Find the area of triangle ABC, given that

(a) $A = 49^\circ$, $b = 21$, and $c = 15$

(b) $a = 12$, $b = 16$, and $c = 5$

Example 8:
Find the area of the region outside a circle of radius 6 feet, but inside a regular heptagon circumscribing the circle.



Example 9:
Polygon triangulation involves partitioning a polygon into non-overlapping triangles using diagonals only. Find the area of an irregular polygon shown below by using polygon triangulation. $AB = 7$, $BC = 10$, $CD = 7.7$, $DE = 6.3$, $EA = 11.2$, $A = 76^\circ$, $C = 36^\circ$.