Chapter 6.6: The Law of Cosines & Area

You're in for a special treat this section.

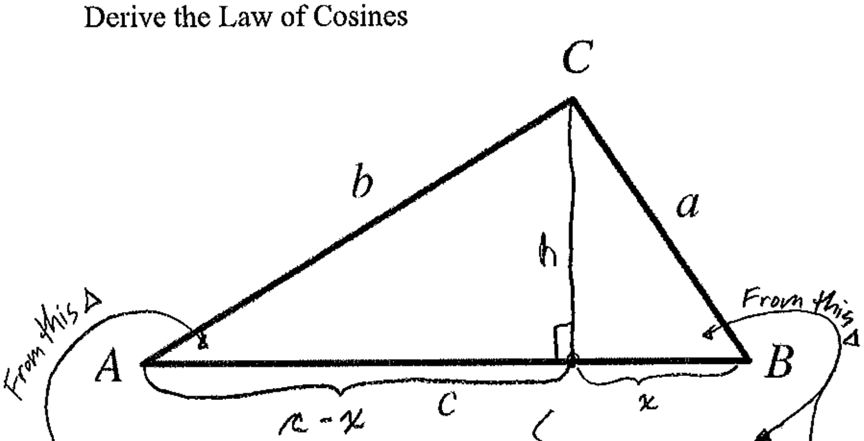
We all know how important, useful, and utterly amazing the Pythagorean Theorem is. Unfortunately, it only works for right triangles.

What if there was a Pythagorean-esque Theorem that worked for ALL triangles? Well, guess what! There is! And it's called the **Law of Cosines.**

Just like the Pythagorean Theorem works well when dealing with side lengths, so does the Law of Cosines, but the LOC also works well when we only have two sides and an included angle.

We will be using the Law of Cosines to solve triangles given **SSS** or **SAS** (although it can also be used in the SSA ambiguous case).

Example 1:
Derive the Law of Cosines



$(c-x)^2+h^2=b^2$

$c^2-2cx+x^2+h^2=b^2$

$x^2+h^2=a^2$

$\cos B=\frac{x}{a}$

So $x=a\cos B$

Substituting right column into left column equation:

$c^2-2c(a\cos B)+a^2=b^2$

rewriting:

$$\boxed{b^2=a^2+c^2-2ac\cos B}$$

* this process can be similarly done using the two other altitudes to get two other versions, summarized at right

Law of Cosines

$$a^2=b^2+c^2-2bc\cos A$$
$$b^2=a^2+c^2-2ac\cos B$$
$$c^2=a^2+b^2-2ab\cos C$$

Super Pythagorean Theorem

The Law of Cosines

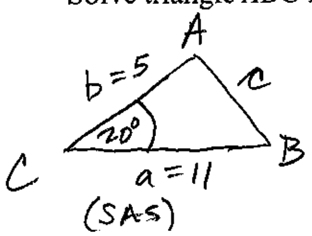
For ANY triangle ABC,

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Example 2: * you must set up Law of Cosines ~~for SAS~~ with given angle on one side and the side opposite the given angle alone on the other side of the equation!

for SAS

Solve triangle ABC if $a = 11$, $b = 5$, $C = 20^\circ$ using the Law of Cosines



$c^2 = 11^2 + 5^2 - 2(5)(11)\cos 20^\circ$ ← the SAS case is the "plug and chug" variety.

$c = \sqrt{11^2 + 5^2 - 2(5)(11)\cos 20^\circ}$ ← we only want the positive root. Why?

$\boxed{c = 6.529}$ ← store as "c" ← problem is now a SSS problem. Find either of two remaining angles with Law of Cosines.

Find A: $11^2 = 5^2 + 6.529...^2 - 2(5)(6.529...)\cos A$

$A = \cos^{-1}\left(\frac{11^2 - 5^2 - 6.529...^2}{-2 * 5 * 6.529...}\right)$

$\boxed{A = 144.816^\circ}$ ← store as A

Find B: $B = 180^\circ - 20^\circ - 144.816^\circ...$

$\boxed{B = 15.183^\circ}$

Be very, very, very careful if you are using the Law of Sines to find Angles (this should ONLY be done in the SSA case, after you've determined the number of solutions). Because and angle and its supplement have the same sine value, you might not always get the correct angle.

Example 3:

Solve triangle ABC if $a = 11$, $b = 5$, $C = 20^\circ$ using the Law of Cosines to find c, then (a) use the Law of SINES to find angle A, then (b) use the Law of Sines to find angle B

From above

Find A: $\frac{\sin A}{11} = \frac{\sin 20^\circ}{6.529...}$

$A = \sin^{-1}\left(\frac{11\sin 20^\circ}{6.529...}\right)$

$\boxed{A = 35.183^\circ}$ ← this is the wrong angle.

a is largest side, but A would ~~not~~ be largest angle!! B would be 124.816°

* if you had used Law of Sines to find angle B, the Smaller of the two remaining angles,

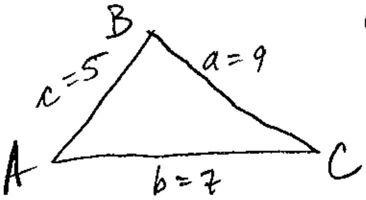
$\frac{\sin B}{5} = \frac{\sin 20^\circ}{6.529...}$

$B = \sin^{-1}\left(\frac{5\sin 20^\circ}{6.529...}\right)$ correct!

$B = 15.183^\circ$

**The Law of Sines CAN be used to find angles in non-ambiguous cases, but care should be taken to find the SMALLER of the two remaining angles. If you're not committed to checking and are desiring a correct answer, use the Law of Cosines.

Example 4:
Solve triangle ABC if $a = 9$, $b = 7$, and $c = 5$ using any methods.



(SSS) choose Any angle to find first. you might want to find the largest angle first so you can use the Law of Sines with reckless abandon afterward.

1st, though, Let's check the Δ inequality: $5+7=12>9$ ✓✓✓

Find A: (largest Angle)

$9^2=5^2+7^2-2*5*7\cos A$

$A=\cos^{-1}\left(\frac{9^2-5^2-7^2}{-2*5*7}\right)$

$\boxed{A=95.739°}$ ← Store as A

Find B:

$\frac{\sin B}{7}=\frac{\sin 95.739°\ldots}{9}$

$B=\sin^{-1}\left(\frac{7\sin 95.739°\ldots}{9}\right)$

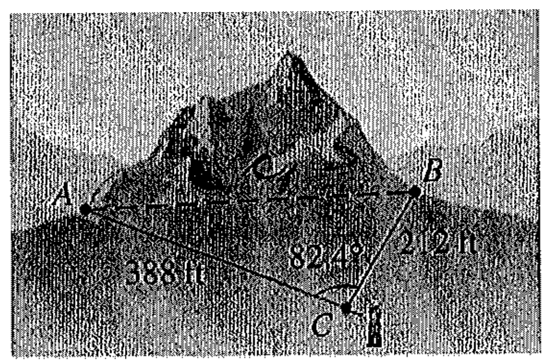
$\boxed{B=50.703°}$ ← Store as B

Find C:

$C=180°-95.739°\ldots-50.703°\ldots$

$\boxed{C=33.557°}$

Example 5:
A tunnel is to be built through a mountain. To determine the approximate length of the tunnel, an ambitious math student with a homemade transit takes to measurements from a fixed point (as shown in the figure). Use the student's data to determine the length of the tunnel.



Let c = tunnel length

$c^2=388^2+212^2-2(388)(212)\cos 82.4°$

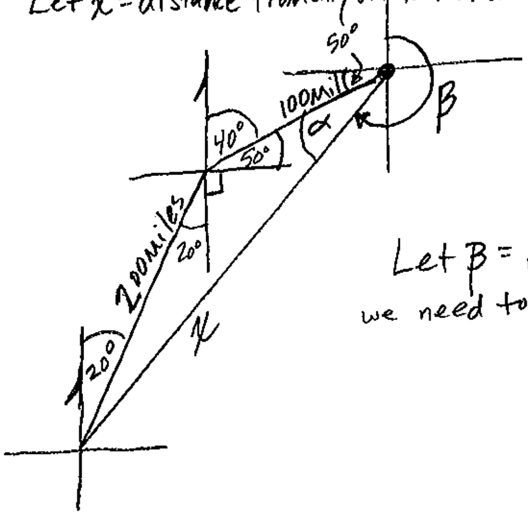
$c=\sqrt{388^2+212^2-2(388)(212)\cos 82.4°}$

$\boxed{c=416.8095929\text{ ft}}$

Example 6:

An airplane pilot sets out from NB Municipal airport on a heading of 20°, flying at 200 miles per hour. After an hour, he changes his heading to 40°. After another half hour on this new heading, he lands at his house. What is his distance from the airport to home. In the morning, when he leaves the house in his plane to return to the airport, on what heading should he fly to get there directly.

Let x = distance from airport to home:



$x^2 = 200^2 + 100^2 - 2*200*100\cos(20^\circ + 90^\circ + 50^\circ)$

$x = \sqrt{200^2 + 100^2 - 2*200*100\cos(160^\circ)}$

$\boxed{x = 295.952 \text{ miles}}$ ← store as X

Let β = Heading from home to airport.

we need to find α: $\frac{\sin\alpha}{200} = \frac{\sin 160^\circ}{295.952\ldots}$

$\alpha = \sin^{-1}\left(\frac{200\sin 160^\circ}{295.952\ldots}\right)$

$\boxed{\alpha = 13.363^\circ\ldots}$ ← store as A

so $\beta = 180^\circ + (90^\circ - 50^\circ - \alpha)$

$\boxed{\beta = 206.636^\circ}$

One last thing regarding triangles that we need to know how to find: Area

Formulas for Areas of Triangles

Right Triangles Area $= \frac{1}{2}bh$	**SAS** Area $= \frac{1}{2}ab\sin C$	**SSS** Area $= \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$
Equilateral Triangles Area$= \frac{\sqrt{3}}{4}s^2$	**ASA or AAS** Area $= \frac{1}{2}a^2\frac{\sin B\sin C}{\sin A}$	

Example 7:

Find the area of triangle ABC, given that

(a) $A = 49^\circ$, $b = 21$, and $c = 15$

(SAS) No repeated letters

Area $= \frac{1}{2}bc\sin A$

Area $= \frac{1}{2}(21)(15)\sin 49^\circ$

$\boxed{\text{Area} = 118.866}$

(b) $a = 12$, $b = 16$, and $c = 5$

$5 + 12 = 17 > 16 \rightarrow \Delta$ exists

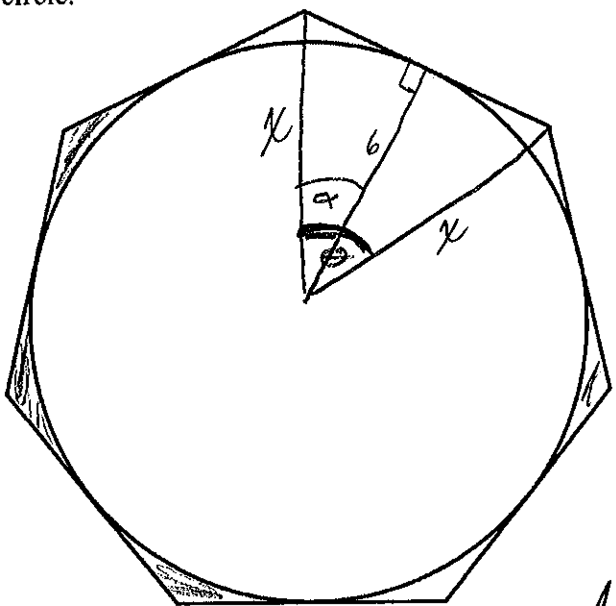
(SSS) s = semiperim $= \frac{12+16+5}{2}$

$s = 16.5$

Area $= \sqrt{16.5(16.5-12)(16.5-16)(16.5-5)}$

$\boxed{\text{Area} = 20.662}$

Example 8:
Find the area of the region outside a circle of radius 6 feet, but inside a regular heptagon inscribed in the circle.



$\theta = \frac{360^\circ}{7}$, $\alpha = \frac{1}{2}\theta = \frac{360^\circ}{14}$

Find x: $\cos\left(\frac{360^\circ}{14}\right) = \frac{6}{x}$, $x = \frac{6}{\cos\left(\frac{360}{14}\right)}$

$\boxed{x = 6.659} \rightarrow x$

Area of Heptagon = 7(Area of 1 triangle with central angle of θ)

$= 7\left[\underbrace{\tfrac{1}{2}(x)(x)\sin\left(\tfrac{360}{7}\right)}_{SAS}\right]$

Area of Hept = $\boxed{121.356\ ft^2} \rightarrow A$

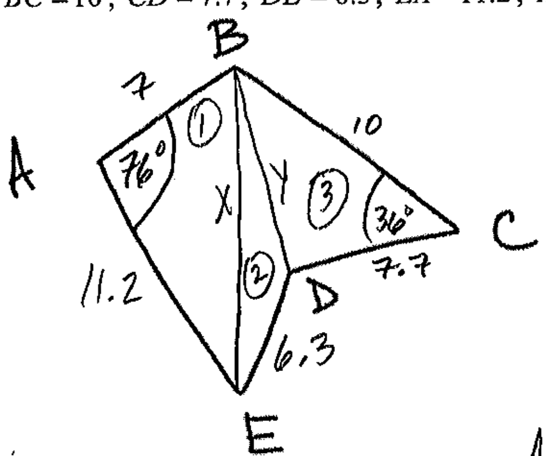
Area of Circle = $\pi(6^2) = \boxed{36\pi\ ft^2}$

Area of region = Area of Hept − Area of Circ

$= A - 36\pi$

$= \boxed{8.259\ ft^2}$

Example 9:
Polygon triangulation involves partitioning a polygon into non-overlapping triangles using diagonals only. Find the area of an irregular polygon shown below by using polygon triangulation. $AB = 7$, $BC = 10$, $CD = 7.7$, $DE = 6.3$, $EA = 11.2$, $A = 76^\circ$, $C = 36^\circ$.



$X = \sqrt{7^2 + 11.2^2 - 2(7)(11.2)\cos 76^\circ} = \boxed{11.683} \rightarrow X$

$Y = \sqrt{10^2 + 7.7^2 - 2(10)(7.7)\cos 36^\circ} = \boxed{5.890} \rightarrow Y$

Area △1 (SAS) $= \frac{1}{2}(7)(11.2)\sin 76^\circ = \boxed{38.035} \rightarrow A$

Area △3 (SAS) $= \frac{1}{2}(10)(7.7)\sin 36^\circ = \boxed{22.629} \rightarrow B$

Area △2 $= \sqrt{D(D-X)(D-Y)(D-6.3)}$

△2 semiperimeter

$= \frac{X + Y + 6.3}{2}$

$= \boxed{11.937} \rightarrow D$