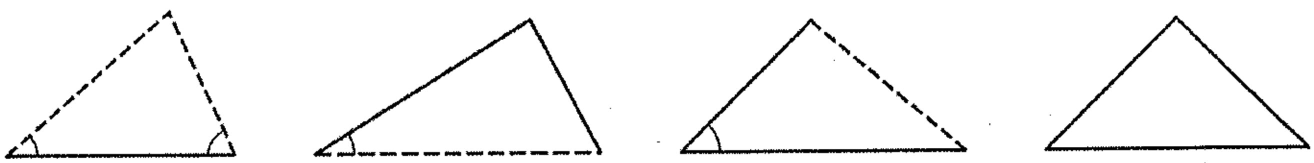
Chapter 6.5: The Law of Sines

SOH CAH TOA and the Pythagorean Theorem are pretty handy, especially if you're dealing with right triangles (not so handy if you're trying to cook a gourmet Ramen feast).

Often, though, we are faced with triangles that aren't right, if not dinner guests with a more sophisticated palette. How can we find missing information quickly and easily for **oblique triangles** (a fancy terms for non-right triangles)?

Recall from geometry that a triangle has six parts: 3 sides and 3 angles. Simply knowing 3 of these 6 will enable us to determine the size and shape of any triangle, right, wrong, or oblique.

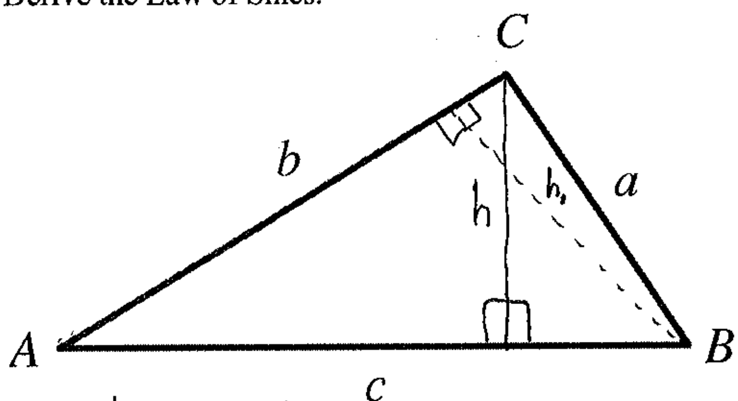


(a) ASA or SAA (b) SSA (c) SAS (d) SSS

The Law of Sines will enable us to find missing pieces of a triangle simply by knowing any of the following:

- Two angles and an adjacent side: AAS
- Two angles and an included side: ASA
- Two sides and an adjacent angle (care must be taken in this case, both in finding the info, and in listing the acronym): SSA

Example 1:
Derive the Law of Sines.



$\sin A = \frac{h}{b}$, $\sin B = \frac{h}{a}$

$h = b\sin A$, $h = a\sin B$

so $b\sin A = a\sin B$

$\frac{b\sin A}{a\cancel{b}} = \frac{\cancel{a}\sin B}{a b}$

$\frac{\sin A}{a} = \frac{\sin B}{b}$

similarly:

$\sin A = \frac{h_1}{c}$, $\sin C = \frac{h_1}{a}$

$h_1 = c\sin A$, $h_1 = a\sin C$

so $c\sin A = a\sin C$

$\frac{\cancel{c}\sin A}{a\cancel{c}} = \frac{\cancel{a}\sin C}{\cancel{a}c}$

$\frac{\sin A}{a} = \frac{\sin C}{c}$

So $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

The Law of Sines

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

(for finding angles, carefully)

or

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

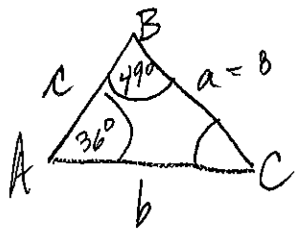
(for finding side lengths with reckless abandon)

Some of the best things about the Law of Sines are that, it's the LAW, and also that it works for ANY triangle. For any triangle, there is a "magical" number that is the same for any of the sides and sines of the corresponding angles

We will start out by using the Law of Sines to find side lengths in the AAS or ASA case. When we find the remaining 3 pieces of a triangle from the 3 given pieces, we say we are **solving the triangle**.

Example 2:

Solve ΔABC if $A=36^\circ$, $B=49^\circ$, and $a=8$



① Find missing angle:

$C=180^\circ-36^\circ-49^\circ$

$\boxed{C=95^\circ}$

② Find b:

$\frac{b}{\sin 49^\circ}=\frac{8}{\sin 36^\circ}$

$b=\frac{8\sin 49^\circ}{\sin 36^\circ}$

$\boxed{b=10.271...}$

③ Find c:

$\frac{c}{\sin 95^\circ}=\frac{8}{\sin 36^\circ}$

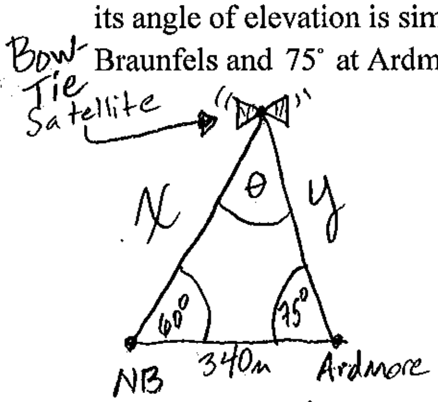
$c=\frac{8\sin 95^\circ}{\sin 36^\circ}$

$\boxed{c=13.558...}$

* a/A is our "pivot" pair
* so $\frac{8}{\sin 36^\circ}$ will be used
* to find b and c

Example 3:

A satellite orbiting the earth passes directly overhead at observation stations in New Braunfels, TX and its step-sister city, Ardmore, OK, 340 miles apart. At an instant when the satellite is between the two stations, its angle of elevation is simultaneously observed, incidentally by two step sisters, to be 60° at New Braunfels and 75° at Ardmore. How far is the satellite from New Braunfels? Ardmore?



*Assume flat distance, rather than curved

$\theta=180^\circ-60^\circ-75^\circ$

$\theta=45^\circ$

Dist from NB

$\frac{x}{\sin 75^\circ}=\frac{340}{\sin 45^\circ}$

$x=\frac{340\sin 75^\circ}{\sin 45^\circ}$

$\boxed{x=464.448 \text{ miles}}$

Dist from Ardmore

$\frac{y}{\sin 60^\circ}=\frac{340}{\sin 45^\circ}$

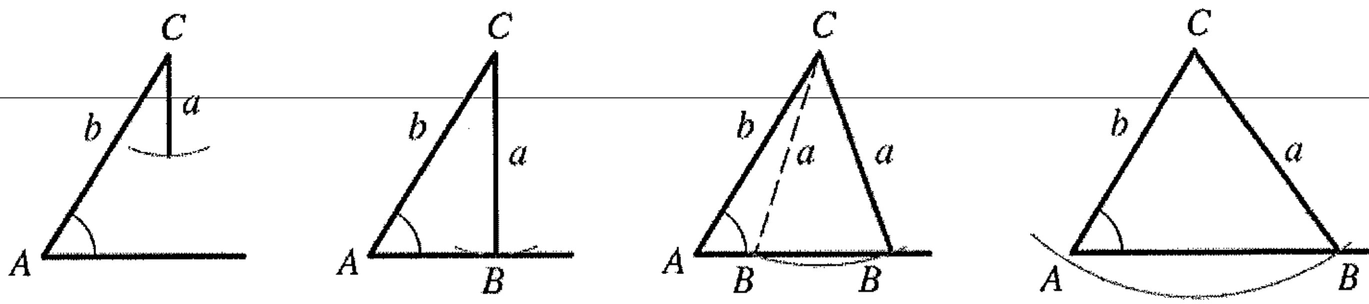
$y=\frac{340\sin 60^\circ}{\sin 45^\circ}$

$\boxed{y=416.413 \text{ miles}}$

The Ambiguous Case

In the SSA case, that is two sides and the NON-inclusive angle, the ambiguous case exists. This information does not necessarily make a unique triangle, and possibly not even a triangle at all. In fact, depending on the proportion of the side lengths to the angle, it could form 0, 1, or 2 triangles. Here's a visual why.

Let's say we're given A, b, and a (notice a repeated letter, in this case Aa).

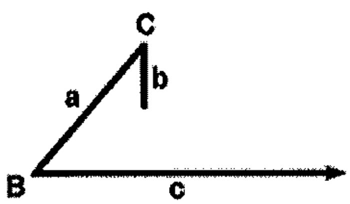


It is helpful to identify the ambiguous case from the beginning of the problem, then draw the information the same way every time to aid in your systematic analysis of how many triangles exist PRIOR to setting out to solve the triangle.

We will now analyze two different cases based on the given angle: The Acute Case and Obtuse Case
Assume we're given B, a, and b:

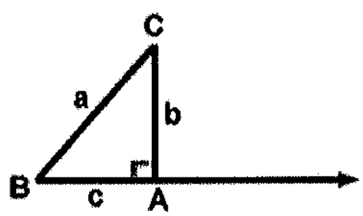
Acute Cases:

Case I:



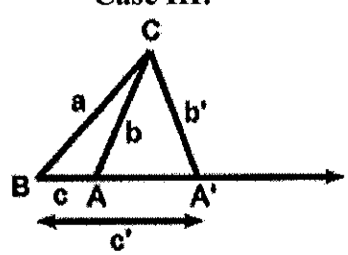
$b < a$, no solution

Case II:



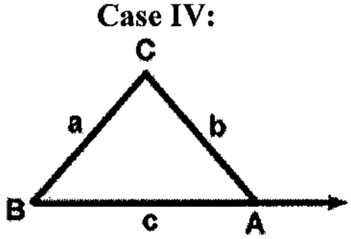
$b = a\sin B$, one right solution

Case III:



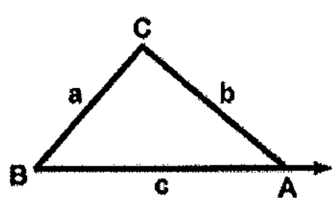
$a\sin B < b < a$, two solutions (one acute, one obtuse)

Case IV:



$b = a$, one isosceles solution

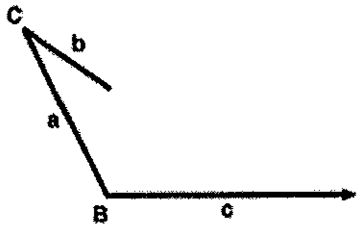
Case V:



$b > a$, one solution

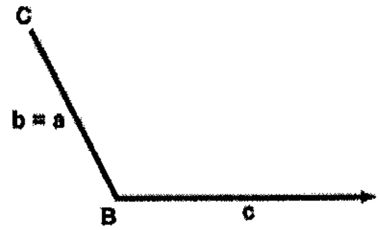
Obtuse Cases:

Case VI:



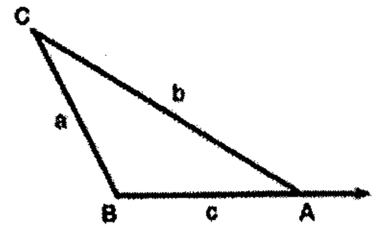
$b < a$, no solution

Case VII:



$b = a$, no solution

Case VIII:



$b > a$, one solution

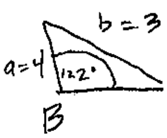
Number of possible triangles given a, b, B

Conditions	$b < a$	$b = a$	$b > a$
B is acute	0, 1, 2 (Cases I, II, III)	1 (Case IV)	1 (Case V)
	0 (Case VI)	0 (Case VII)	1 (Case VIII)

Example 4:

Determine the number of triangles ABC formed by the given information:

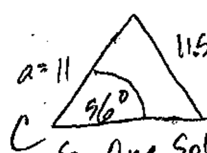
(a) $a = 4$, $b = 3$, $B = 122^\circ$



$3 < 4$ and $122^\circ > 90^\circ$

so No Solution

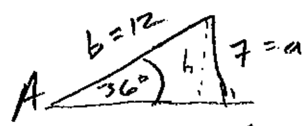
(b) $C = 56^\circ$, $a = 11$, $c = 11.5$



$11.5 > 11$

So One Solution

(c) $b = 12$, $a = 7$, $A = 36^\circ$

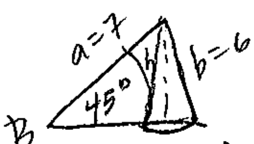


$h = 12\sin 36^\circ = 7.0534$

$a < 7.0534 \rightarrow$ too short

So No solution

(d) $a = 7$, $b = 6$, $B = 45^\circ$

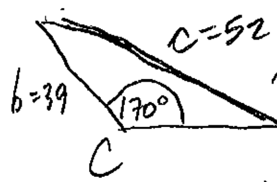


$h = 7\sin 45^\circ = 4.949$

$h < 6 < a$

so 2 solutions

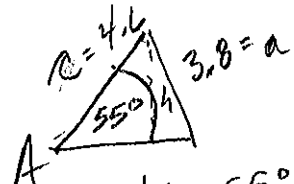
(e) $b = 39$, $c = 52$, $C = 170^\circ$



$c > b$

So one solution

(f) $a = 3.8$, $c = 4.6$, $A = 55^\circ$



$h = 4.6\sin 55^\circ = 3.768$

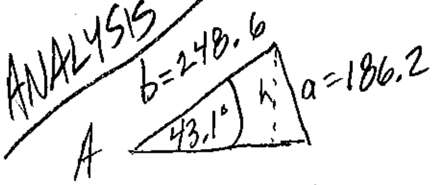
$h < 3.8 < c$

So 2 solutions

Example 5:

Solve triangle ABC if $A = 43.1°$, $a = 186.2$, and $b = 248.6$.

ANALYSIS



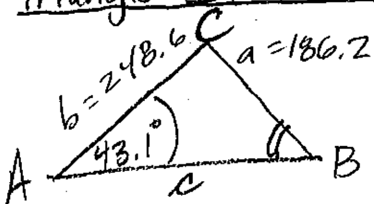
$h = 248.6 \sin 43.1°$

$h = 169.861$

$h < 186.2 < b$

so 2 triangles

Triangle 1 (acute)



$\dfrac{\sin 43.1°}{186.2} = \dfrac{\sin B}{248.6}$

$\sin B = \dfrac{248.6 \sin 43.1°}{186.2}$

$B = \sin^{-1}\left(\dfrac{248.6 \sin 43.1°}{186.2}\right)$

This will always give you a positive acute angle because of the PVR of $f(x) = \arcsin x$!

$\boxed{B = 65.818°}$ (store as B)

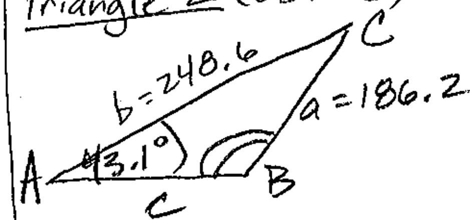
$C = 180° - 43.1° - 65.818...$

$\boxed{C = 71.081°}$ (store as C)

$\dfrac{c}{\sin 71.081°...} = \dfrac{186.2}{\sin 43.1°}$

$\boxed{c = 257.790}$

Triangle 2 (obtuse)



** the NEW angle B is the supplement of the previous B!!

$B = 180° - 65.818844...$

$\boxed{B = 114.181°}$ (store as new B)

$C = 180° - 43.1° - 114.1811...$

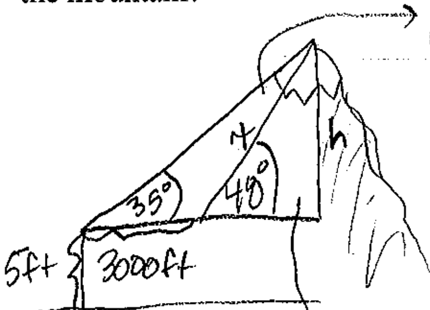
$\boxed{C = 22.718°}$ (store as new C)

$\dfrac{c}{\sin 22.718...} = \dfrac{186.2}{\sin 43.1°}$

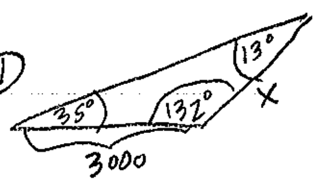
$\boxed{c = 105.246}$

Example 6:

To measure the height of a mountain, a survey takes two sightings of the peak at a distance 3000 feet apart at the same elevation. The first observation results in an angle of elevation of 48° and the second results in an angle of elevation of 35°. If the transit he used to measure the angles is 5 feet tall, what is the height of the mountain?



①

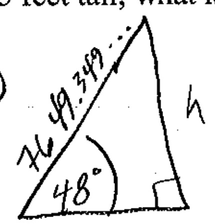


$\dfrac{x}{\sin 35°} = \dfrac{3000}{\sin 13°}$

$x = \dfrac{3000 \sin 35°}{\sin 13°}$

$x = 7649.349...\text{ ft}$

②



$\sin 48° = \dfrac{h}{7649.349...}$

$h = (7649.349...) \sin 48°$

$h = 5684.574\text{ ft}$

So Mountain is h + 5ft stand

$\boxed{\text{Mountain is } 5689.574 \text{ feet tall}}$