Chapter 6.4: Other Identities

The first new set of identities is a direct consequence of the sum composite identities.

Double Angle Identities

$$\sin 2x = 2\sin x\cos x \qquad \cos 2x = \begin{cases} \cos^2 x - \sin^2 x \\ 2\cos^2 x - 1 \\ 1 - 2\sin^2 x \end{cases} \qquad \tan 2x = \frac{\sin 2x}{\cos 2x} = \frac{2\tan x}{1-\tan^2 x}$$

Example 1:
Prove the following double-angle identity:

$$\cos 2x \stackrel{?}{=} 2\cos^2 x - 1$$

$\cos(x+x)$

$\cos x\cos x - \sin x\sin x$

$\cos^2 x - (\sin^2 x)$

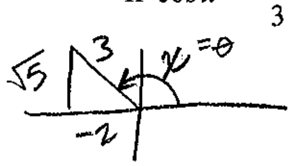
$\cos^2 x - (1-\cos^2 x)$

$\cos^2 x - 1 + \cos^2 x$

$2\cos^2 x - 1$

– Sizzle

Example 2:
If $\cos x = -\frac{2}{3}$ and $\sin x > 0$, find $\cos 2x$, $\sin 2x$, and $\tan 2x$. (Verify on calculator)



$\cos 2x$
$= \cos^2 x - \sin^2 x$
$= \left(-\frac{2}{3}\right)^2 - \left(\frac{\sqrt{5}}{3}\right)^2$
$= \frac{4}{9} - \frac{5}{9}$
$= \boxed{-\frac{1}{9}}$

$\sin 2x$
$= 2\sin x\cos x$
$= 2\left(\frac{\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right)$
$= \boxed{-\frac{4\sqrt{5}}{9}}$

$\tan 2x$
$= \frac{\sin 2x}{\cos 2x}$
$= \frac{-\frac{4\sqrt{5}}{9}}{-\frac{1}{9}}$
$= \cancel{-}\frac{4\sqrt{5}}{\cancel{9}} \cdot \left(\cancel{-}\frac{\cancel{9}}{1}\right)$
$= \boxed{4\sqrt{5}}$

$x = \cos^{-1}\left(-\frac{2}{3}\right) = 2.3005\ldots$

$2x = 4.60104\ldots$ rad

$\cos(4.601\ldots) = -0.111\ldots$
$= -\frac{1}{9}$ ✓

$\sin(4.601\ldots) = -.993\ldots$
$= -\frac{4\sqrt{5}}{9}$ ✓

$\tan(4.601\ldots) = 8.944\ldots$
$= 4\sqrt{5}$ ✓

Example 3:
Write $\cos 3x$ in terms of $\cos x$.

$\cos 3x$

$\cos(2x+x)$

$(\cos 2x)\cos x-(\sin 2x)\sin x$

$(2\cos^2 x-1)\cos x-(2\sin x\cos x)\sin x$

$2\cos^3 x-\cos x-2(\sin^2 x)\cos x$

$2\cos^3 x-\cos x-2(1-\cos^2 x)\cos x$

$2\cos^3 x-\cos x-2\cos x+2\cos^3 x$

$4\cos^3 x-3\cos x$

The previous example shows that $\cos 3x$ can be written as a third-degree polynomial in terms of $\cos x$. The identity $\cos 2x = 2\cos^2 x - 1$ shows that $\cos 2x$ can be written as second-degree polynomial in terms of $\cos x$. In general, for any positive integer n, we can write $\cos nx$ as an n-degree polynomial in tems of $\cos x$. A similar result for $\sin nx$ is not generally true.

When proving an identity in which the anlges on one side are different than those on the other, it should be a priority to get the angles the same by using identities.

Example 4:
Prove the following identity:

$$\frac{\sin 3x}{\sin x\cos x} \overset{?}{=} 4\cos x - \sec x$$

Left side:

$$\frac{\sin(2x+x)}{\sin x\cos x}$$

$$\frac{(\sin 2x)\cos x+\sin x(\cos 2x)}{\sin x\cos x}$$

$$\frac{(2\sin x\cos x)\cos x+\sin x(2\cos^2 x-1)}{\sin x\cos x}$$

$$\frac{\cancel{\sin x}\left[2\cos^2 x+2\cos^2 x-1\right]}{\cancel{\sin x}\cdot\cos x}$$

$$\frac{4\cos^2 x-1}{\cos x}$$

Right side:

$$\frac{4\cos x}{1}-\frac{1}{\cos x}$$

$$\frac{4\cos^2 x-1}{\cos x}$$

–Nuttin to it!

In calculus, it is important to be able to write even powers of sine and cosine in terms of single powers of cosine. The following power-reducing identities are derived from the double-angle cosine identities.

Power-Reducing Identities

$$\sin^2 x=\frac{1}{2}(1-\cos 2x) \qquad \cos^2 x=\frac{1}{2}(1+\cos 2x) \qquad \tan^2 x=\frac{1-\cos 2x}{1+\cos 2x}$$

Example 5:

Express $\sin^2 x\cos^2 x$ in terms of a single power of cosine.

$\frac{1}{2}(1-\cos 2x)\cdot\frac{1}{2}(1+\cos 2x)$

$\frac{1}{4}(1-\cos 2x)(1+\cos 2x)$

$\frac{1}{4}(1-\cos^2 2x)$

$\frac{1}{4}-\frac{1}{4}\cos^2 2x$

$\frac{1}{4}-\frac{1}{4}\left(\frac{1}{2}(1+\cos 4x)\right)$

$\frac{1}{4}-\frac{1}{8}(1+\cos 4x)$

$\frac{1}{4}-\frac{1}{8}+\frac{1}{8}\cos 4x$

$\boxed{\frac{1}{8}-\frac{1}{8}\cos 4x}$

Solving the power-reducing identites for the individual trig functions allows us to find exact trig values for angles that are half of our Unit Circle angles. Notice the the "price" for reducing the power, is to double the angle.

Half-Angle Identities

$$\sin\left(\frac{1}{2}x\right)=\pm\sqrt{\frac{1-\cos x}{2}} \qquad \cos\left(\frac{1}{2}x\right)=\pm\sqrt{\frac{1+\cos x}{2}} \qquad \tan\left(\frac{1}{2}x\right)=\pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

The sign of the radical depends on the quandrant in which the half-angle $\frac{1}{2}x$ terminates.

Example 6:

Find the exact value of $\sin\frac{\pi}{8}$ and $\cos\frac{7\pi}{8}$ using the half-angle identites.

$\sin\frac{\pi}{8}=\sin\left(\frac{1}{2}\left(\frac{\pi}{4}\right)\right)$

($\frac{\pi}{8}$ terminates in QI where sine is POSITIVE)

$=+\sqrt{\frac{1}{2}\left(1-\cos\frac{\pi}{4}\right)}$

$=\sqrt{\frac{1}{2}\left(1-\frac{\sqrt{2}}{2}\right)}$

$=\sqrt{\frac{1}{2}-\frac{\sqrt{2}}{4}}$

$=\sqrt{\frac{2-\sqrt{2}}{4}}=\boxed{\frac{\sqrt{2-\sqrt{2}}}{2}}$

$\cos\frac{7\pi}{8}=\cos\left(\frac{1}{2}\left(\frac{7\pi}{4}\right)\right)$

($\frac{7\pi}{8}$ terminates in QII where cosine is negative)

$=-\sqrt{\frac{1}{2}\left(1+\cos\frac{7\pi}{4}\right)}$

$=-\sqrt{\frac{1}{2}\left(1+\frac{\sqrt{2}}{2}\right)}$

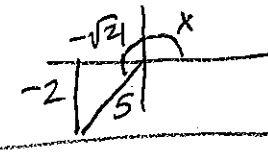
$-\sqrt{\frac{1}{2}+\frac{\sqrt{2}}{4}}$

$-\sqrt{\frac{2+\sqrt{2}}{4}}=\boxed{-\frac{\sqrt{2+\sqrt{2}}}{2}}$

Example 7:

Find $\tan\left(\frac{x}{2}\right)$ if $\sin x = -\frac{2}{5}$ and $x \in \left(\pi, \frac{3\pi}{2}\right)$. if $\pi < x < \frac{3\pi}{2}$, then $\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$,

So $\frac{1}{2}x$ terminates in QII, where tangent is neg



$$\tan\left(\frac{x}{2}\right) = -\sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$= -\sqrt{\frac{1-\left(-\frac{\sqrt{21}}{5}\right)}{1+\left(-\frac{\sqrt{21}}{5}\right)}}$$

$$= -\sqrt{\frac{1+\frac{\sqrt{21}}{5}}{1-\frac{\sqrt{21}}{5}}\left(\frac{5}{5}\right)}$$

$$= -\sqrt{\left(\frac{5+\sqrt{21}}{5-\sqrt{21}}\right)\left(\frac{5+\sqrt{21}}{5+\sqrt{21}}\right)}$$

$$= -\sqrt{\frac{25+10\sqrt{21}+21}{25-21}}$$

$$= -\sqrt{\frac{46+10\sqrt{21}}{4}}$$

$$= \boxed{-\frac{1}{2}\sqrt{46+10\sqrt{21}}}$$

CHECK: $\sin^{-1}\left(-\frac{2}{5}\right) = -.4115 =$ ref angle

So ref angle is $0.4115\ldots$

$x = \pi + 0.4115\ldots = 3.553\ldots$ rad

$\tan\left(\frac{3.553\ldots}{2}\right) = -4.791\ldots = -\frac{1}{2}\sqrt{46+10\sqrt{21}}$ ✓

Example 8:

Solve algebraically in the interval $[0, 2\pi)$: $\sin 2x = \cos x$

$\sin 2x = \cos x$

$2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

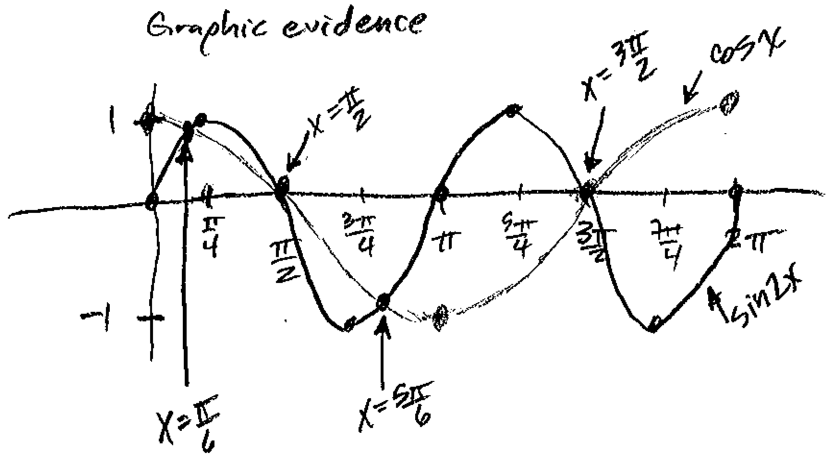
$\cos x = 0$ or $\sin x = \frac{1}{2}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad$ $x = \frac{\pi}{6}, \frac{5\pi}{6}$

All in domain, so

$$\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}}$$

Graphic evidence



Example 9:

Solve: $\sin^2 x = 2\sin^2\left(\frac{x}{2}\right)$. Find all solutions in radians.

$\sin^2 x = 2\left(\frac{1}{2}\left(1-\cos\left(2\left(\frac{x}{2}\right)\right)\right)\right)$

$1-\cos^2 x = 1 - \cos x$

$0 = \cos^2 x - \cos x$

$\cos x(\cos x - 1) = 0$

$\cos x = 0$ or $\cos x = 1$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad$ $x = 0$

So
$$\boxed{\begin{aligned} x &= \frac{\pi}{2} + \pi n, n \in \mathbb{Z} \\ x &= 0 + 2\pi n, n \in \mathbb{Z} \end{aligned}}$$

Example 10:

Solve the following equations in the interval $x \in [0, 2\pi)$.

(a) $2\sin 3x - 1 = 0$

$\sin 3x = \frac{1}{2}$

$3x = \sin^{-1}\left(\frac{1}{2}\right)$

$$\begin{cases} 3x = \frac{\pi}{6} + 2\pi n \\ 3x = \frac{5\pi}{6} + 2\pi n \end{cases}$$

$$\begin{cases} x = \frac{\pi}{18} + \frac{2\pi}{3} n \\ x = \frac{5\pi}{18} + \frac{2\pi}{3} n \end{cases}$$

General Solutions

So $\boxed{x = \frac{\pi}{18}, \frac{13\pi}{18}, \frac{25\pi}{18} \quad x = \frac{5\pi}{18}, \frac{17\pi}{18}, \frac{29\pi}{18}}$

(b) $2\cos\left(\frac{x}{3}\right) + \sqrt{3} = 0$

$2\cos\left(\frac{x}{3}\right) = -\sqrt{3}$

$\cos\left(\frac{x}{3}\right) = -\frac{\sqrt{3}}{2}$

$\frac{x}{3} = \cos^{-1}\left(-\sqrt{3}/2\right)$

$$\begin{cases} \frac{x}{3} = \frac{5\pi}{6} + 2\pi n \\ \frac{x}{3} = \frac{7\pi}{6} + 2\pi n \end{cases}$$

$$\begin{cases} x = \frac{5\pi}{2} + 6\pi n \\ x = \frac{7\pi}{2} + 6\pi n \end{cases}$$

General Solutions

both $\frac{5\pi}{2}$ and $\frac{7\pi}{2} > 2\pi$
So No Solutions
on $[0, 2\pi)$

(c) $\sin 2x = \sin 4x$

$\sin 2x - \sin 4x = 0$

$\sin 2x - \sin(2(2x)) = 0$

$\sin 2x - 2\sin 2x \cos 2x = 0$

$\sin 2x(1 - 2\cos 2x) = 0$

$\sin 2x = 0$ or $\cos 2x = \frac{1}{2}$

$$\{2x = 0 + \pi n \qquad \{x = \frac{\pi}{2} n$$

$$\begin{cases} 2x = \frac{\pi}{3} + 2\pi n \\ 2x = \frac{5\pi}{3} + 2\pi n \end{cases}$$

$$\begin{cases} x = \frac{\pi}{6} + \pi n \\ x = \frac{5\pi}{6} + \pi n \end{cases}$$

$\boxed{x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \quad x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}}$