Chapter 5.3: Circular Trigonometric Functions

Definition

A **reference triangle** is formed by "dropping" a perpendicular (altitude) from the terminal ray of a standard position angle to the *x*-axis, that is, again, the *x*-axis. The **reference angle** will be the positive, acute angle of the reference triangle between the terminal ray and the *x*-axis.

Reference triangles are used to find trigonometric values for their standard position angles. They are of particular importance for standard position angles whose terminal sides reside in Quadrants II, III, or IV.

Example 1:
Draw a reference triangle for an angle θ that terminates in the following quadrants. Label the reference angle and the reference triangle. Describe mathematically how to find the reference angle in each case in terms of both degrees and radians.
(a) Quadrant I (b) Quadrant II (c) Quadrant III (d) Quadrant IV

Definition

A **trigonometric function** is a ratio of 2 of 3 sides of a right triangle formed by drawing a **reference triangle** with reference angle θ_{ref} from an independent angle θ in standard position.

Example 2:
Draw a reference triangle in Quadrant I, dropping your perpendicular from the point (x, y) on the terminal ray. Label the hypotenuse r, then list all the possible ratios of x, y, and r.

Definition

Let θ be any real angle, and let (x, y) be the terminal point from which the perpendicular is dropped creating a reference triangle with hypotenuse r. Then we define the six ratios of the side lengths of the reference triangle to be the following

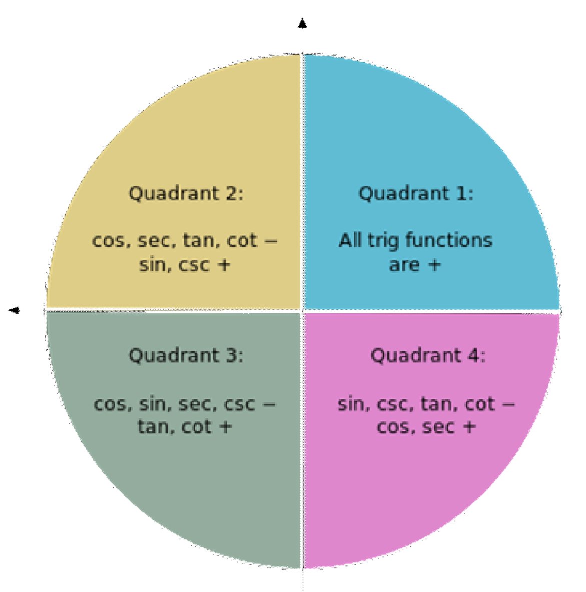
$\sin\theta = \frac{y}{r}$ (sine function)	$\cos\theta = \frac{x}{r}$ (cosine function)	$\tan\theta = \frac{y}{x}$ (tangent function)
$\csc\theta = \frac{r}{y}$ (cosecant function)	$\sec\theta = \frac{r}{x}$ (secant function)	$\cot\theta = \frac{x}{y}$ (cotangent function)

Because these functions can be defined by rotating any radius *r* through any angle θ in standard position, they are referred to as **circular trigonometric functions**.

Example 3:
If $\sin\theta = \frac{5}{6}$ and $90^\circ < \theta < 180^\circ$, find the simplified, exact value of the other five trig functions of θ. Find the value of θ and θ_{ref} using the calculator.

Depending on which quadrant an angle θ terminates, the sign of each of the six trig functions can be either positive or negative. Because *r* will end up being the **radius of rotation**, it is **always positive**. Therefore the signs of the trig functions are determined exclusively by the signs of *x* and *y*.

The chart at right show these signs.



Example 4:

If $\tan\theta = \dfrac{5}{12}$ and $\csc\theta < 0$, determine the simplified, exact value of the other five trig functions of θ.
Find the value of $\theta \in \left[0^\circ, 360^\circ\right)$ and θ_{ref}.

Example 5:

If $\cot\theta = -2$ and $\cos\theta > 0$, determine the simplified, exact value of the other five trig functions of θ.
Find the value of $\theta \in \left[0^\circ, 360^\circ\right)$ and θ_{ref}.

Example 6:

If the terminal side of θ passes through the point $(-4,3)$, find the simplified, exact values of all six trig functions of θ.

Definition

A **quadrantal angle** is an angle that terminates on either the *x*- or *y*- axis. Quadrantal angles have no relevant reference angles since in each case the reference triangle either collapses vertically or horizontally.

Example 7:
For a circle of radius 1 unit centered at the origin, find the value of the six trig functions for each of the following quadrantal angles:
(a) 0° (b) 90° (c) 180° (d) 270° (e) 360° (f) 720° (e) 1080°

The circle mentioned in the previous example is called a **Unit Circle**. Reference angles of 30°, 45°, and 60° show up quite often in calculations. Consequently, it is worth developing the cosine and sine values for all the angles within one positive rotation around the Unit Circle. This will be LOTS of FUN!

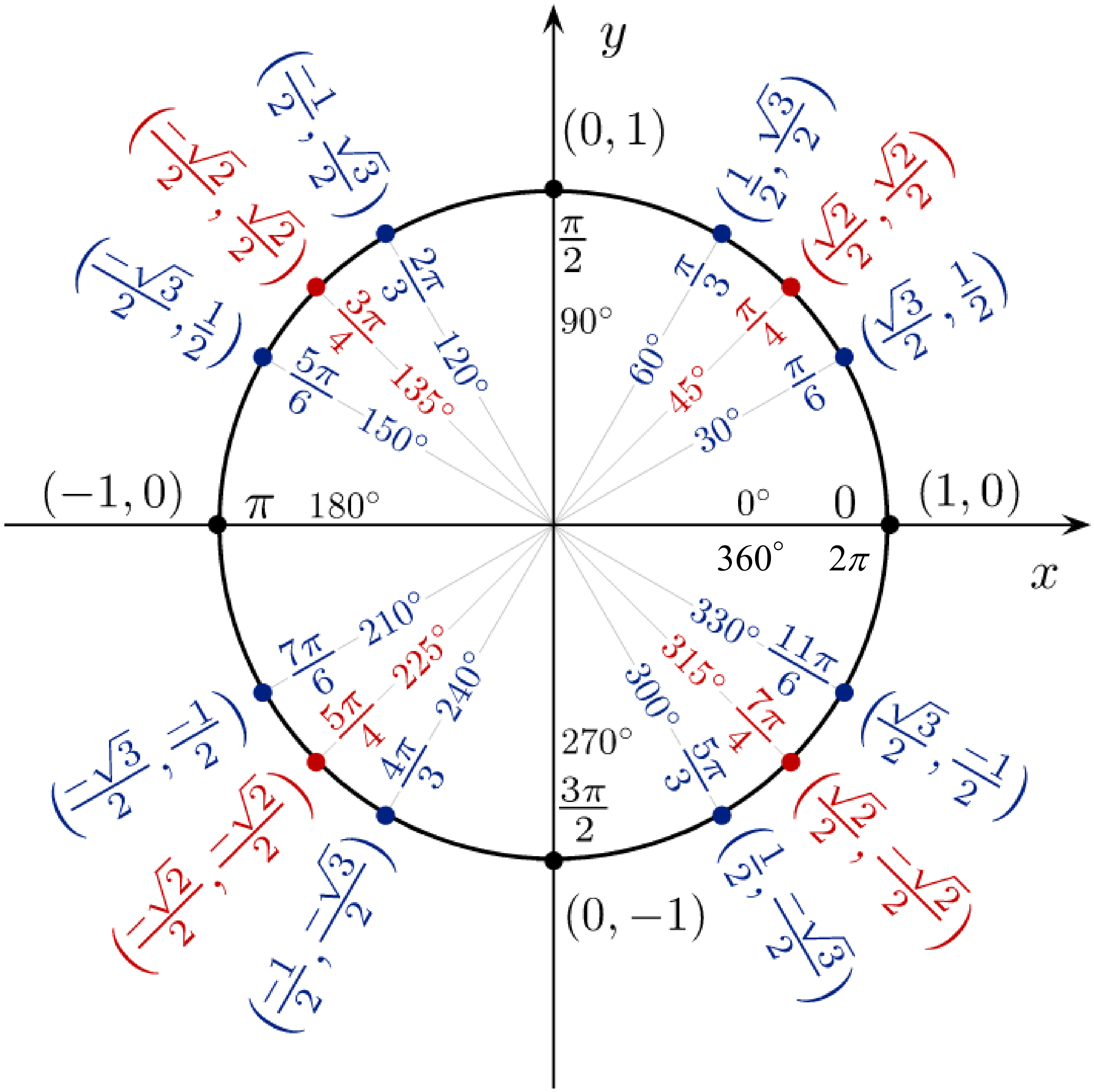
Before we can do that, though, we must review two special triangles from geometry.

Example 8:
Draw a $30^\circ - 60^\circ - 90^\circ$ and a $45^\circ - 45^\circ - 90^\circ$ triangle. For each, scale the hypotenuse to be one unit long. Then find the following:

(a) $\cos 30^\circ$ (b) $\sin 30^\circ$ (c) $\cos 45^\circ$

(d) $\sin 45^\circ$ (e) $\cos 60^\circ$ (f) $\sin 60^\circ$

We will now develop the Unit Circle.



Each coordinate (x, y) on the unit circle not only represents a point on the circumference of the circle, but, more importantly, represents the cosine and sine values, repectively of the angle θ in standard position. That is

$$(x, y) = (\cos\theta, \sin\theta)$$

From only these two trig functions, we can obtain the other four by using the following **trigonometric identities** *(An identity is an equation that is true for all values of the variable in the domain of each expression.)*

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

*Note: These Unit Circle ratios work regardless of the size of the circle or triangle. Since the scale factor affects all three sides, it will always divide out in the ratios.

Example 9:
Using the Unit Circle and the identities, find the six trig functions for the following angles.

(a) $\theta = \frac{5\pi}{4}$ (b) $\theta = \frac{11\pi}{6}$ (c) $\theta = \frac{3\pi}{4}$ (d) $\theta = \frac{3\pi}{2}$

Theorem

Coterminal angles have the same trig ratios.

Example 10:
Find the simplified, exact value of the following using the Unit Circle.

(a) $\sin\left(-210^\circ\right)$ (b) $\tan\frac{16\pi}{3}$ (c) $\cos\left(205{,}155^\circ\right)$ (d) $\csc 3655\pi$

Not all angles are on the Unit Circle. For these angles, if we want to approximate their trig ratios, we must use the calculator exclusively.

Example 11:
Evaluate the following on the calculator. Be sure you are in the correct "mode" on your calculator. Report three decimals.

(a) $\sin 327^\circ 15' 12''$ (b) $\sec 45$ (c) $\cot 201.53^\circ$ (c) $\csc\frac{19\pi}{5}$

Note: Although we have been measuring angles in both degrees and radians, from this point forward, we will be using radians, since these are real numbers and unitless.