Chapter 5.1: Angles and their Measures

The study of angles and their measure dates waaaay back to the ancient world. Engineers and astronomers needed to measure angles to construct their structures and to study the heavens. **Hipparchus of Nicaea** (190-120 B.C.), an ancient astronomer, is known as the father of trigonometry for compiling the first set of trigonometric tables to simplify the study of celestial movement. Be sure to thank Hipparchus when you later get to trig proofs.

With the eventual demands of global navigation and the increasing need for accurate maps of large areas, trigonometry evolved into a full-fledged, important branch of mathematics. In fact, the word "trigonometry" didn't emerge until the late 16th century when **Bartholomaeus Pitiscus** first used the word as the title of his published work *Trigonometria* (1595).

So what is an angle and how do we measure it?

Definition

An **angle** is **the measure of rotation** between two rays sharing a vertex.

In geometry, when you first encountered trig, you measured angles in degrees, using the degree bubble, $^\circ$, to denote the **specific units** of degrees. But what is a degree anyway?

Definition

A degree is a unit of angular measure that is equal to $\dfrac{1}{360}$ of a full rotation.

This means that there are 360° in one full rotation/revolution. Why 360°? That seems arbitrary. In fact, assigning degree-measure to angles, itself, is somewhat arbitrary. It has something to do with the Ancient Babylonians, who used a base 60, or **sexagesimal**, number system. 60 was a "sexy" number for their base because it was evenly divisible by so many whole numbers. There are few other numbers as small as 360 that have so many different factors. This makes the degree a very nice unit to divide the circle into an equal number of

> Some believe that the Babylonians divided the circle into 360 parts because they thought that it took 360 days for the Sun to go around the Earth.

parts. 120° is 1/3 of a circle, 90° degrees is 1/4, and so on.
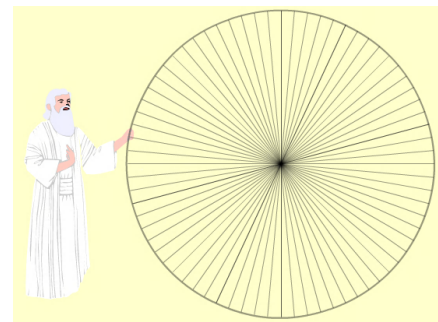
In our modern culture, there is another carry-over from the Babylonian sexagesimal number system—in the way we tell time: 1 hour is 60 minutes, 1 minute is 60 seconds, 1 second is 60 hours (in a boring math class).

We usually measure fractions of degrees in **decimal degrees** (base 10). But there's another way. In the same way we measure fractions of hours in minutes and seconds, so too we can measure fractions of degrees in **minutes** (units are ') and **seconds** (units are "). Maps and GPS coordinates use these.
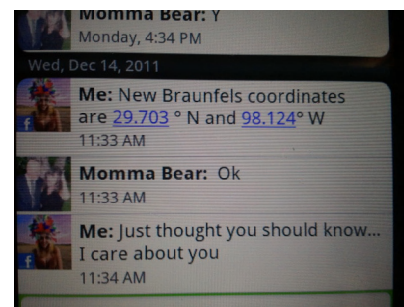
Example 1:
Draw an angle of $76.425°$, then convert it to DMS (degree, minutes, seconds)

Example 2:
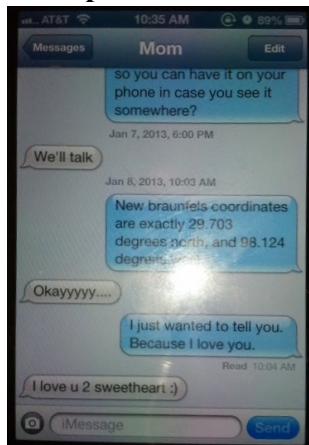
New Braunfels, Texas lies at $29°42'10"N / 98°7'27"W$. Convert the GPS coordinates of our city into decimal degree coordinates. Text your results to your parents.


In geometry, as in Example 1, your angles were not specifically oriented. In other words, a given angle and its connecting rays could be rotated, flipped, translated, etc. without any loss of generalization. Additionally, the angles were always positive. In trigonometry, we're interested in analyzing the geometry, so we must standardize the process by superimposing a coordinate grid on our angles. This will allow us to us x's and y's and talk about equations, functions, graphs, and other fun stuff. Additionally, angles will now be vector quantities, meaning they can be positive or negative, which will imply the direction of rotation. Moreover, angles will be able to take on such magnitudes of unimaginable quantities that they'll make your head spin!

Definition

An angle in **standard position** has its initial ray on the positive *x*-axis. The terminal ray may land anywhere. A **counterclockwise** rotation will result in a **positive** angle (this is why the quadrants are numbered the way they are.) A **clockwise** rotation will result in a **negative** angle. We use such Greek letters as θ, ϕ, α, and β to denote both the name and size of the angle.

The pictures below show some angles in standard position. What are some plausible angle measures for each?
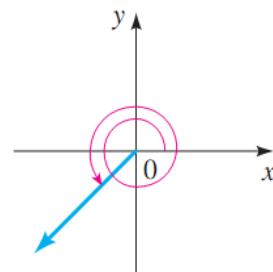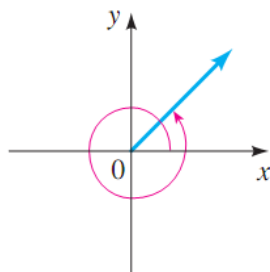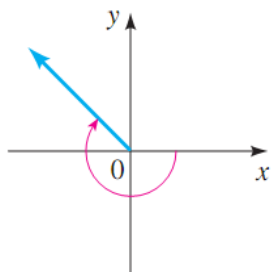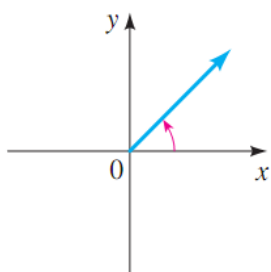
Example 3:
Draw the following angles in standard position. Show the arc between the initial and terminal ray.

(a) $\theta = 190°$ (b) $\beta = -276°$ (c) $\phi = -91°$ (d) $\alpha = 629°$ (e) $\theta = 1822°$ (f) $\Omega = -57631°$

In Example 3, the angles $\phi = -91°$ and $\alpha = 629°$ shared a terminal ray. Although we cannot say these angles are equal, we can say they are something else.

Definition

Two angles are **coterminal** with each other if they share a terminal ray. In general, any given angle θ, its coterminal angles are given by $\theta_{cot} = \theta + 360°n$ where $n \in \mathbb{Z}$.

Example 4:
If $\theta = 465°$, find all the angles, θ_{cot}, coterminal with θ for $-1000° < \theta_{cot} < 1000°$

Example 5:
For each angle ϕ, find the angle, θ, coterminal with ϕ such that $0° \leq \theta < 360°$

(a) $\phi = -28549°$ (b) $\phi = 45678910°$

In addition to coterminal angles, there is another special angle that is associated with each and every angle in standard form.

Definition

A **reference angle**, θ_{ref} , is a **positive**, **acute** angle that measure the angle between the terminal ray and the closest x-axis.

For any angle that terminates in one of the four quadrants, there are three other angles within one rotation that have the same reference angle. All coterminal angles have the same reference angle.

Example 6:
If $\beta = -265.363°$, (a) draw β in standard form (b) find the reference angle β_{ref} (c) find the coterminal angle β_{cot} to β that is 47 positive rotations away (d) convert β , β_{ref} , and β_{cot} into DMS notation (round to the nearest second).
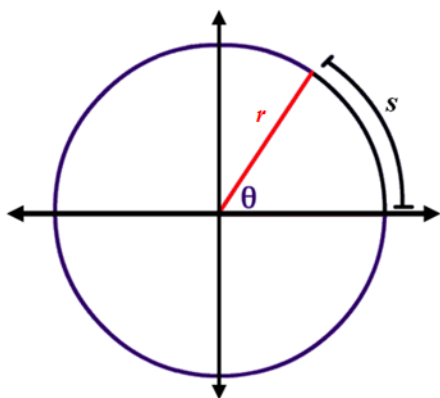
Aside from decimal degrees and DMS, there is a 3rd, more preferable way, to measure angles called a **radian**.

Radian measure of an angle is not an arbitrary number chosen by man, but is rather based on the basic components of a circle.

In a circle of **radius r** drawn on the coordinate plane with its center at the origin, an angle θ drawn within the circle will intercept the circumference of a circle creating an **arc of length s**, where θ is called the **central angle**.

Definition

A **radian** is the measure of the central angle of a circle subtended by an arc equal in length to the radius of a circle. A **radian measure of an angle**, θ , in standard position is the ratio of the arc length, s, to the radius, r, of the circle centered at the origin. This can be represented mathematically as

$$\theta = \frac{s}{r}$$
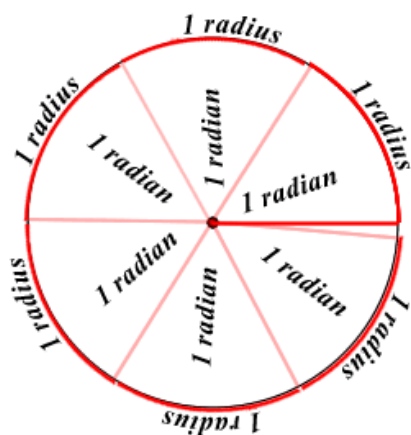
The definition of the radian measure of an angle ensures us that the units will divide out. Consequently, the radian is a **unitless measure** of rotation, whereas degrees are unit-specific. Additionally, the radian is irrespective of the size of a circle/radius. ***When we use radians to represent angle measures, we will write "rads" simply to describe the angle measure. Remember though, that "rads" are not the units!!!!***

The diagram at right shows that for any given circle, its radius can be wrapped around its circumference a little more than six full times. We will now be more exact in determining the radian measure of a full rotation.

Example 7:
Using the equation for the circumference of a circle with radius *r* and the definition of the radian measure of an angle, determine the exact and approximate radian measure of a full rotation. Using your result, develop a conversion factor between degrees and radians. Use your new conversion factor to determine the number of decimal degrees one radian is.

Converting between degrees and radians

When converting between degrees and radians, use the following conversion factors

$$\frac{\pi}{180°} \quad \text{or} \quad \frac{180°}{\pi}$$

Example 8:
Convert the following angles from degrees to radians. Leave π visible in your answer.

(a) $90°$ (b) $180°$ (c) $270°$ (d) $360°$ (e) $60°$

(f) $30°$ (g) $15°$ (h) $45°$ (i) $-175.4°$ (j) $\dfrac{5\pi}{4}^{\circ}$

Example 9:

Convert the following angles from radians to decimal degrees.

(a) $\dfrac{5\pi}{4}$ (b) $\dfrac{11\pi}{6}$ (c) $\dfrac{2\pi}{3}$ (d) $\dfrac{7\pi}{5}$ (e) 1.346

Example 10:

For each of the following angles θ, find the coterminal angles α such that $\alpha \in [0, 2\pi)$. Then, find the reference angle β for each angle.

(a) $\theta = \dfrac{46365\pi}{7}$ (b) $\theta = -\dfrac{789637\pi}{11}$ (c) $\theta = 14998763\pi$ (d) $\theta = 36142.51$