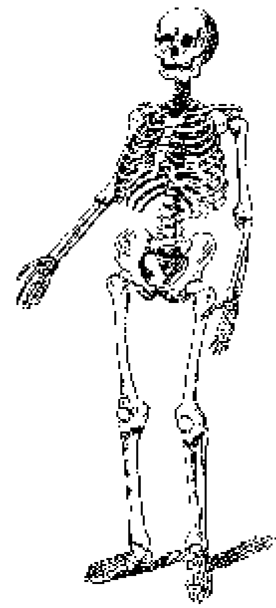
Chapter 4.3: Logarithmic Functions



Logs shouldn't be scary. If you don't like them, that's okay. But they are nothing to be feared (or hated, that's such a strong emotion.)

Logs were invented by John Napier (originally called "Napier's Bones"—definitely more scary). When we began looking to the stars and under the microscope, we began looking at very large and very small numbers. Napier knew we could look at the exponents of such numbers to refer to the relative size of these numbers. Alas, logs were born.

So what, exactly, is a log?? Well, a log is simply an exponent (reread). It is a unique exponent to which one must raise a particular base to achieve a particular result.

For example, since, $2^4 = 16$, it follows that we must raise the base 2 to the 4th power to achieve 16. Thusly, we can say that $\log_2 16 = 4$.

Think of a log function as "undoing" the work of an exponential. In the previous illustration, if we plug in a 4, we get out a 16. In the inverse function, the log function, we can plug in a 16 and get back our original 4.

Every exponential function of the form $f(x) = b^x$, with $b > 0$, $b \neq 1$, is a one-to-one function, and therefore has an inverse function. This inverse function f^{-1} is called the **logarithmic function** with base *b* and is denoted by $f^{-1}(x) = \log_b x$. Recall that for two functions that are inverses of each other: $f^{-1}(x) = y$ if and only if $f(y) = x$ This leads to the following "conversion formula" relating a log to an exponential. Remember, if it's in a box, it's very important.

Conversion Formula

$$\log_b x = y \iff \text{(iff) } b^y = x$$

Where *b* is the base, *y* is the exponent or log, and *x* is called the **argument**.

Example 1:
Solve for *x* in each of the following:

(a) $\log_3 27 = x$ (b) $x = \log_4 \frac{1}{64}$ (c) $\log_2 \sqrt{32} = x$

Fact: Since $f(x)=b^x$ and $g(x)=\log_b x$ are one to one, if $b^x=b^y$, then $x=y$, and if $\log_b x=\log_b y$, then $x=y$.

Additional Fact: The previous fact is in a box, while this one is not.

There are two special logarithmic bases we will look at, both of which are pre-programmed on your calculator: base 10 and base *e*.

- The logarithm with base 10 is called the common logarithm and is denoted by omitting the base: $\log x=\log_{10} x$
- The logarithm with base *e* is called the natural logarithm and is denoted by $\ln$: $\ln x=\log_e x$ (from the Latin *logarithmus naturalis*)

Properties of Common Logs	Properties of Natural Logs	Properties of Logs base *b*
• $\log 1=0$ • $\log 10=1$ • $\log 10^x=x$ • $10^{\log x}=x$	• $\ln 1=0$ • $\ln e=1$ • $\ln e^x=x$ • $e^{\ln x}=x$	• $\log_b 1=0$ • $\log_b b=1$ • $\log_b b^x=x$ • $b^{\log_b x}=x$

Example 2:
Evaluate the following:

(a) $\log_7 7^{3x^2+2}$ (b) $\ln e^x$ (c) $5^{\log_5 5x}$ (d) $e^{\ln x}$ (e) $\log_{22} 22^x$ (f) $57^{\log_{57} x}$

Example 3:
Evaluate the following:

(a) $\log_5 1$ (b) $\log_{1/2} 1$ (c) $\log_b 1$ (d) $\log_{12} 12$ (e) $\log_b b$ (f) $\ln e$ (g) $\log 10$

Summary:

- $\log_b y=x \Leftrightarrow b^x=y$
- $\log_b 1=0$
- $\log_b b=1$
- $\log_b b^x=x=b^{\log_b x}$

As I already mentioned, our TI calculators are preprogrammed to handle logs base *e* and base 10. Find the LN and LOG buttons on your calculator. Now look at the second function these keys represent. Coincidence??? I think not!!!

So what if we're trying to evaluate a log of a different base? Well, we can consult the archaic log tables, or we can use the **change of base formula** with our calculators.

Change of Base Formula

$$\log_b x = \frac{\ln x}{\ln b} = \frac{\log x}{\log b} = \frac{\log_a x}{\log_a b}$$

Example 4:
Evaluate the following on the calculator (think about a reasonable answer before you go to the calculator):

(a) $\log_{0.5} 14$ (b) $\log_6$ ® (c) $\log_{3.6} 17$

Example 5:
Simplify the following using the change of base formula

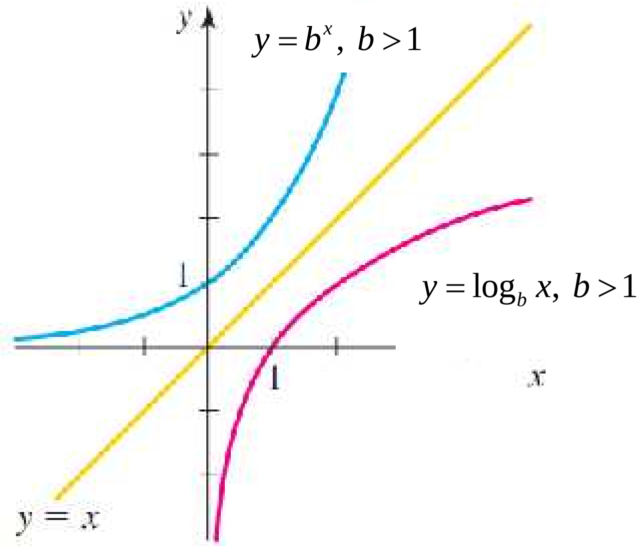
$$(\log_3 5)(\log_2 7)(\log_7 3)(\log_5 11)$$

We can now solve logarithmic equations with variables in all kinds of crazy places. Often, converting a logarithmic equation to exponential can be helpful. Remembering that both exponential and logs are one-to-one will also be helpful.

Example 6:
Solve the following equations for *z*:

(a) $\log z = 3$ (b) $\log_2(z+1) = 4$ (c) $\log_{1/2}(2z+3) = 5$ (d) $\log_z 25 = 2$ (e) $z = \log_5 \frac{5}{125}$

It's time to look at the graphs of logarithmic functions in general.



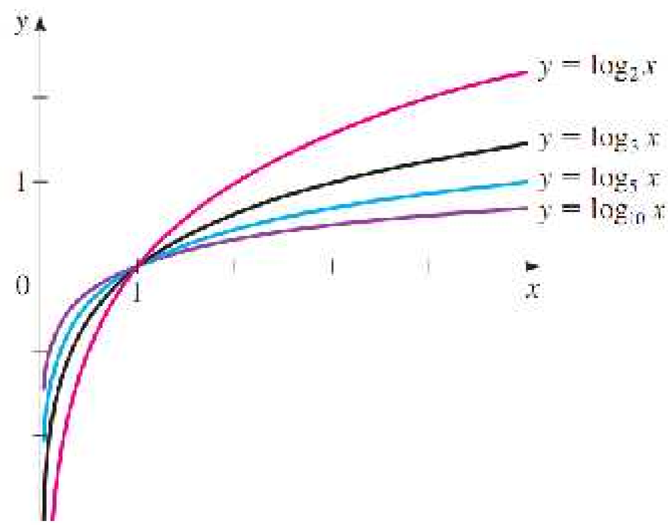
Recall that if a function, like $y = b^x$, is one-to-one, then it has a unique inverse function whose graph can be obtained by reflecting the graph of the original function across the line $y = x$.

This function will have the same base as the exponential and will have an equation of $y = \log_b x$.

If $b > 1$, the log function will be increasing.
If $0 < b < 1$, the log function will be decreasing.

Every log function of the form $y = \log_b x$ will have a vertical asymptote at $x = 0$, an x-intercept at $(1, 0)$, a domain of $(0, \infty)$, and a range of $(-\infty, \infty)$.

For exponential functions, the larger the base, the st
steep the graph, the smaller the base, the steeper the graph. Where would $y = \ln x$ fit into the family of log functions below??



Example 7:
Sketch the following functions. Show the asymptote, *y*-intercept, find the domain and range, then determine any end-behavior.

(a) $f(x) = -2\log_3(x+4)$ (b) $A(x) = 1 + 3\log_{0.5}(4 - x)$

Example 8:
Find the domain of the following functions:

(a) $f(x) = 3 - 4\ln\left(x^2 - 9\right)$

(b) $g(x) = 2 + \log\left(2 - \sqrt{x}\right)$

Example 9:
For $h(x) = 3\log_4\left(\log_5 x\right) + 8$

(a) Find the domain of $h(x)$

(b) $h^{-1}(x)$

Example 10:
If $f(x) = 2\log_{\frac{1}{2}}(2x - 4) + 3$, find

(a) the Domain of $f(x)$

(b) $f^{-1}(x)$

(c) the Domain of $f^{-1}(x)$

(d) the Range of $f(x)$