Chapter 4.2: Exponential & Logistic Modeling

For real-life applications, our independent variable is usually time, t.

Example 1:
Start with an initial value of \$100. Assuming this amount increases by 30% each year, write a model to predict the amount A after t years.

Example 2:
Start with an initial value of \$100. Assuming this amount decreases by 30% each year, write a model to predict the amount A after t years.

Exponential Growth/Decay Model

If an amount A is changing at a constant percentage rate R each time period, t, then

$$A(t) = A_0(1+r)^t$$

Where A_0 is the initial amount and r is R expressed as a decimal.

- If $A(t)$ is exponential growth, then R is the **growth rate**. $1+r$ is called the **growth factor**.
- If $A(t)$ is exponential decay, then R is the **decay rate**. $1-r$ is called the **decay factor**.

Example 3:
Determine the growth/decay rates and factors for each of the following:

(a) $A(t) = 6 \cdot 1.046^t$ (b) $M(t) = 3(.14159)^t$ (c) $B(x) = 7 \cdot 2^x$

Example 4:
Tell whether each of the population models, is an exponential growth function or exponential decay function, and find the growth/decay rates and factors of each. Use the model to predict the current population, then compare this number to the actual current population. Are they the same? Why or Why not??

(a) San Jose, CA: $P(t) = 782,248 \cdot 1.0136^t$
where $t = 0$ corresponds to 1990.

(b) Detroit, MI: $P(t) = 1,203,368 \cdot 0.9858^t$
where $t = 0$ corresponds to 1980.

Exponential growth and decay models are appropriate for modeling any type of behavior where the growth or decay rate is directly proportional to the current amount.

Example 5:
A small culture of 10 bacteria is placed into a petri dish. Supposing that the culture of bacteria doubles every hour, after how many hours will the number of bacteria reach 500,000?

Example 6:
What if the initial bacteria culture from example 5 tripled every two hours? After how many hours would it take for the population to reach 500,000? What would the model be, with t in hours, if the population tripled every (a) 7 hours? (b) day? (c) 20 minutes?

Example 7:

Determine the exponential function of the form $y = A \cdot b^t$ that satisfies the following conditions. Define the units you will use for t.

(a) Initial value = 7, Increasing at a rate of 32% per day

(b) Initial value = 427, Decreasing at a rate of 0.427% per year

(c) Initial value = 18, quadrupling every 13 minutes

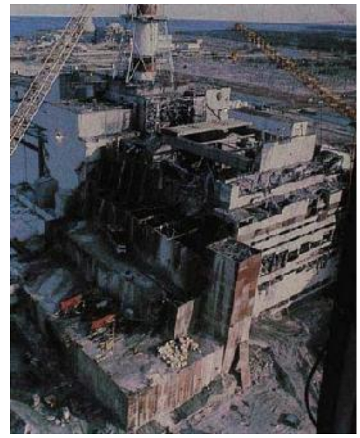
(d) Initial value = 26.3, halving every 3 fortnights (One fortnight is 2 weeks)

Many elements are unstable as are many isotopes of stable elements. These elements and isotopes undergo **radioactive decay**, losing a fixed fraction of their mass per unit of time. The time it takes for these samples to lose half of their mass is called the **half-life** of the substance.

Example 8:
Ra-225, Radium 225, a radioactive isotope of Ra-226, has a half-life of approximately 15 days. If we start with a 6 gram sample of Ra-225, (a) how much of the original sample is left after 12 weeks? (b) How many days will it take the original sample to decay to 1 gram? *Note: Ra-226 is the most stable atomic weight, but still radioactive, with a half-life of 1600 years.

Example 9:
Suppose a radioactive sample loses 43% of its mass every 9876 years. (a) What percentage of its mass will be left after 2000 years? (b) After how many years will the sample contain 22% of its original mass?

Example 10:



On April 26, 1986, the Number Four reactor at the nuclear power plant at Chernobyl, Ukraine exploded releasing over 100 radioactive elements into the atmosphere. Most of these were short lived and decayed quickly. Iodine, strontium, and cesium were the most dangerous of the elements released, and have half-lives of 8 days, 29 years, and 30 years respectively. The isotopes Strontium-90 and Cesium-137 are therefore still present in the area to this day. While iodine is linked to thyroid cancer, Strontium can lead to leukemia. Cesium is the element that travelled the farthest and lasts the longest. This element affects the entire body and especially can harm the liver and spleen. The reactor disaster in Fukushima, Japan (March 11, 2011) rivaled Chernobyl's levels. (a) If 2000 grams of Cesium-137 were released at Chernobyl, how many grams of Cesium-137 are present there today? (b) If 20 grams is considered a "safe" level, how many years will it take for the levels of Cesium-137 to reach this level?

Remember, for modified exponential growth called, **logistic**, the rate of growth is jointly proportional to the current amount and the difference between the current amount and the carrying capacity.

Example 11:

Write an equation of a Logistic Growth model of the form $f(x)=\dfrac{L}{1+Ce^{-kx}}$ with the following characteristics: $f(0)=20$, $f(3)=120$, $\lim_{x\to\infty} f(x)=500$

Example 12:
In 2009, NBHS, with about 2200 students, closed its doors because of the swine flu. Assuming the flu spread throughout the campus according to the logistic equation

$$S(t) = \frac{2200}{1+999e^{-0.1t}},$$

where S is the number of students who have been infected of t days, where $t = 0$ is the day the flu first begins to spread.

(a) How many students on campus were originally infected? At the end of day 3? Day 10?
(b) After how many days have 10 students been infected?
(c) After how many days is the flu spreading at the fastest rate?
(d) According **the model**, after how many days will the entire student population become infected? How can you tell?
(e) If the school policy is to shut down school if at least 5% of the student population is infected, after how many days will the school shut down?