Chapter 3.5: Rational Functions

A rational number is a ratio of two integers. A **rational function** is a quotient of two polynomials. All rational numbers are, therefore, rational functions as well.

Let's get reacquainted with an old friend.

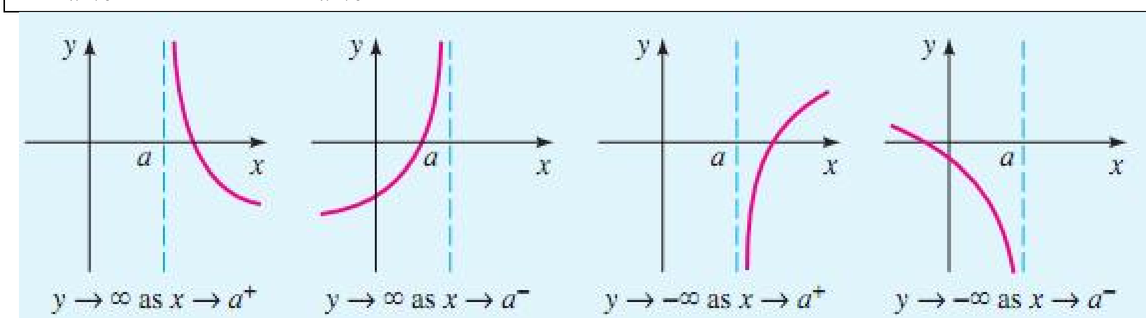
Example 1:

Sketch $f(x)=\frac{1}{x}$. Find the domain and range. Find and label all discontinuities. Find the intervals over which the function is increasing and decreasing. Describe any symmetry. Evaluate the following:

$\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$ (c) $\lim_{x\to 0} f(x)$ (d) $\lim_{x\to 4} f(x)$ (e) $\lim_{x\to -\infty} f(x)$ (f) $\lim_{x\to \infty} f(x)$

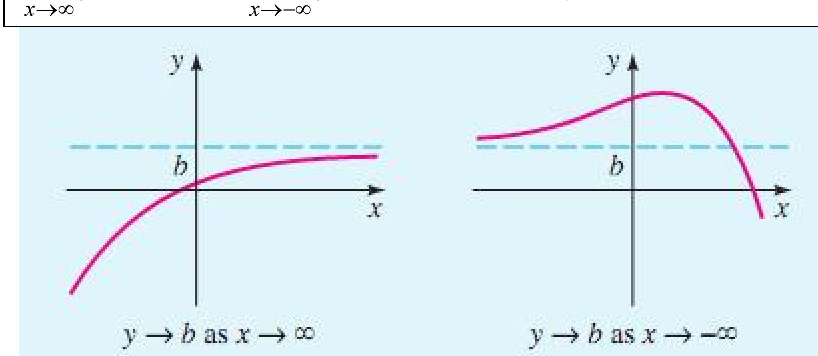
Definition of a **Vertical Asymptote**

If $\lim_{x\to c^+} f(x)=\pm\infty$ or $\lim_{x\to c^-} f(x)=\pm\infty$, then there exists a **vertical asymptote** at $x=c$.



Definition of a **Horizontal Asymptote**

$\lim_{x\to\infty} f(x)=L$ **or** $\lim_{x\to -\infty} f(x)=L$ if and only if there exists a **Horizontal Asymptote** (HA) at $y=L$



Example 2:

Sketch a graph of each rational function by a transformation of the parent function $y = \frac{1}{x}$. Identify the domain and range, all asymptotes, and all discontinuities.

(a) $f(x) = \frac{6}{4-2x}$

(b) $g(x) = \frac{3x-5}{x+2}$

Example 3:

Sketch the graph of each of the following functions. Identify the domain and range, all asymptotes, and all discontinuities.

a) $r(x) = \frac{5x+21}{x^2+10x+25}$

b) $R(x) = \frac{x^2-3x-4}{2x^2+4x}$

c) $f(x) = \frac{4x^2-28x+48}{3x^3+3x^2-36x}$

Asymptotic and Discontinuous behavior of Rational Functions

Let R be the rational function

$$R(x)=\frac{a_n x^n + a_{n-1}x^{n-1}+\cdots+a_1x+a_0}{b_m x^m + b_{m-1}x^{m-1}+\cdots+b_1x+b_0}$$

1. The Vertical Asymptotes (non-removable infinite discontinuities) of R are the lines $x=a$, where a is a zero of the denominator but NOT the numerator. That is $R(a)=\frac{\neq 0}{0}$.
2. The Holes (removable point discontinuities) of R are the points $\left(b, \lim_{x\to b} R(x)\right)$, where b is a zero of BOTH the numerator and denominator. That is $R(b)=\frac{0}{0}$
3. NOTE: If $R(c)=\frac{0}{\neq 0}=0$, then $x=c$ is a zero/x-intercept/root of $R(x)$.
4. (a) if $n<m$, then $\lim_{x\to\infty} R(x)=0$ and R has an HA at $y=0$.

   (b) if $n=m$, then $\lim_{x\to\infty} R(x)=\frac{a_n}{b_m}$, and R has an HA at $y=\frac{a_n}{b_m}$

   (c) if $n>m$, then $\lim_{x\to\infty} R(x)=\infty$ or $\lim_{x\to\infty} R(x)=-\infty$, and R has no HA.

A horizontal asymptote is an example of an **end-behavior model**. There are other types of end-behavior models that can be found the same way—analyzing the leading coefficients in the numerator and denominator. The behavior of the end-behavior model and the original function will be the same as $x\to\infty$ and as $x\to-\infty$, although the **local behavior** (for small x-values) will be different.

Example 4:

Identify the leading term in the end behavior model of the following rational functions. Based on the end-behavior model, determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

(a) $f(x)=\frac{4x^2-2x+11x^3+4}{7-4x}$ (b) $f(x)=\frac{3x^3-4x^5-\pi x^2+4x}{3x-11x^2+26.2}$ (c) $f(x)=\frac{85x^{99}+38x^{47}+8x^{33}+4x}{77x^4+1492x^{88}-5x^{98}-111}$

Example 5:
Find the domain, end behavior, and all discontinuities. Sketch the function. Find the range. Use long division to find the equation of the end-behavior model. Verify each on the calculator, then zoom out to see the end behavior.

(a) $f\left(x\right)=\frac{x^3-16x}{2x^2+6x-8}$

(b) $f\left(x\right)=\frac{\left(x^2-1\right)\left(x^2-3x+3\right)}{x^2-3x+2}$

Example 6:
Analyze the graphs of the following rational functions:

(a) $f\left(x\right)=\frac{2x^2-18}{x^2-4}$

(b) $h\left(x\right)=\frac{x^2+x}{x}$

(c) $j\left(k\right)=\frac{2x^2-2}{x^2-3x+2}$

(d) $p\left(x\right)=\frac{x}{x^2-3x}$

(e) $Q\left(x\right)=\frac{2x^3-18x}{x^3-4x}$

(f) $g\left(x\right)=\frac{\left(x^2-2x+4\right)\left(x-6\right)}{x^2-8x+12}$

Example 7:

Construct the equation (in factored form) of a holey graph with holes at $x=1$, $x=5$, and $x=-4$, *x*-intercepts at $x=2$ and $x=-6$, a vertical asymptotes at $x=3$ and $x=6$ with a horizontal asymptote at $y=\frac{2}{3}$.

Here's a quick summary of how to analyze rational functions:

1. **Factor**: Factor both the numerator and denominator
2. **Domain**: Find the values that make the denominator zero. This will be domain restrictions.
3. **Discontinuities**: $\frac{\neq 0}{0}$ means "VA." $\frac{0}{0}$ means "hole."
4. **Bad Guy**: Divide out any "bad guy" factors causing a hole. Use the equation that remains for all further analysis, <u>including the *y*-value of the hole.</u>
5. **End Behavior**: Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Graph all HA's and SA's.
6. **Intercepts**: Find the *x*-intercepts by determining the zeros of the numerator, and the *y*-intercept from the value of the function at $x=0$.
7. **Symmetry**: even, odd or neither
8. **Sketch the Graph**: Graph all asymptotes first, intercepts next, then combine the other information to fill in the rest of the graph.
9. **Smile**: Pat yourself on the back for a job well done!

Example 8

Graph the rational function $m(x)=\dfrac{5x^2+21x}{x^3+10x^2+25x}$

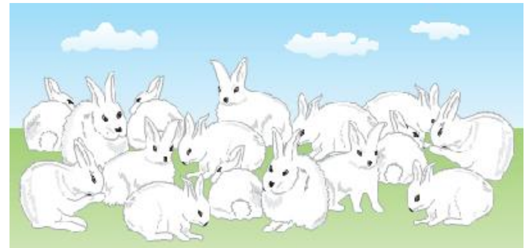
Example 9:

Graph the rational function $L(x)=\dfrac{(x^2-4x-5)(x+2)}{x^2-x-6}$

Example 10:

Write an equation of a function with a VA at $x=-\dfrac{1}{2}$, an SA at $y=-3x-2$, a y-intercept at 4, and a hole at $x=500$. Find the end behaviors of this function. As $x\to\infty$, what do the slopes of the function approach?

Example 11:



Suppose that the rabbit population on Mr. Korpi's Hairy Hare farm follows the formula

$$P(t) = \frac{3000t}{t+1}$$

Where $t \geq 0$ is the time (in months) since the beginning of the year.

(a) Draw a graph of the rabbit population on the relevant domain.
(b) According the model, what is the initial population of the rabbits? How can this be??
(c) Using a calculator, what will the rabbit population be after 5.5 months?
(d) Using a calculator, after how many months will the rabbit population be at 1066 rabbits?
(e) According to the math model, what eventually happens to the rabbit population, in the long run?