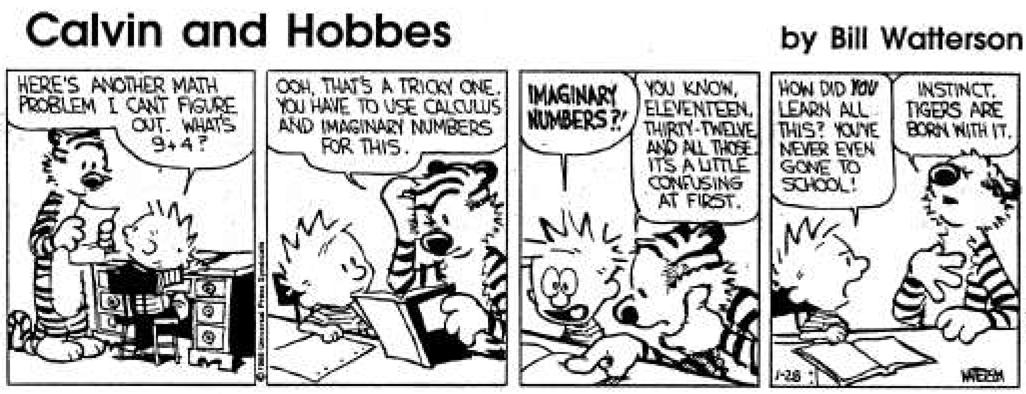
Chapter 3.4: Complex Zeros of Polynomials



Imaginary numbers were first encountered in the first century in ancient Greece when Heron of Alexandria came across the square root of a negative number in his calculation for a truncated pyramid. He fudged the math and moved on. They didn't begin to establish a foothold until Italian algebraists began solving cubic and higher-order equations. **Scipione dal Ferro** made some progress in this field, but he still overlooked the complex roots ("impossible" solutions, he called them) lurking in his equations. Instead, it was **Girolamo Cardano** in 1584 who first paved the way for the acceptance of BOTH negative and imaginary numbers. Later, **Rene Descartes** referred to these types of numbers as "**imaginary**", and he meant it as a derogatory term. It wasn't until **Euler** (in 1777 gave us the symbol ***i*** to equal $\sqrt{-1}$) and **Gauss** that imaginary numbers, and the complex number system, gained acceptance.

Today, the world wouldn't be the same without these "imaginary" numbers. Key fields as quantum mechanics and electromagnetism depend on the mathematics of imaginary numbers. When engineers design airplane wings or cell-phone towers, imaginary numbers are vital to their calculations.

In 1799, Carl Friedrich Gauss, at the tender age of 22, earned his first doctorate degree with a proof that is fundamental to our study of algebra. It established a theorem that was so dear to him that he went on to write three additional, different proofs of the theorem over his lifetime, the fourth and final proof coming fifty years after the first. So what was the theorem?

The Fundamental Theorem of Algebra

A polynomial function of degree *n* has exactly *n* complex roots (including multiplicities).

Example 1:

Find all the complex roots of (a) $f(x) = x^2 - 1$ and (b) $f(x) = x^2 + 1$

Before we set out to explore the implications of this theorem or find more of these complex roots, we need to practice our ability to handle calculations involving imaginary numbers and explore what they really are.

Example 2:

If $i = \sqrt{-1}$, complete the following table. Use your results to simplify i^{26457}.

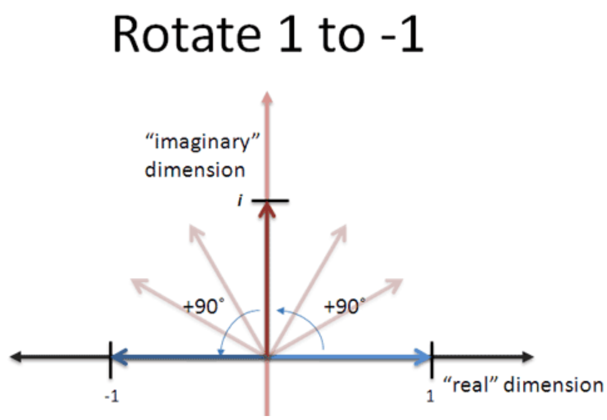
$i^{-2} =$
$i^{-1} =$
$i^{0} =$
$i = \sqrt{-1}$
$i^{2} =$
$i^{3} =$
$i^{4} =$
$i^{5} =$
$i^{6} =$

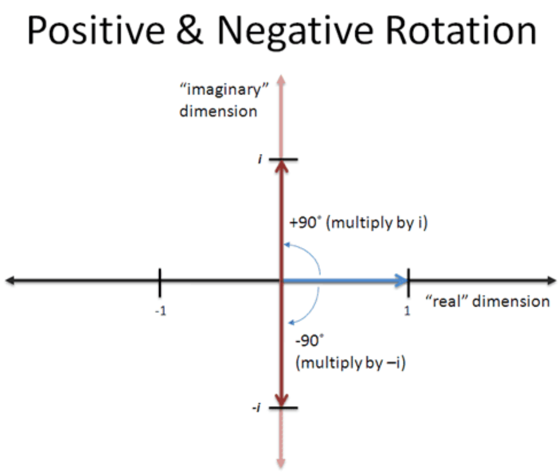
Why does this work? Here's brief explanation.

When we say $x^2 = -1$, we really mean $1 \cdot x^2 = -1$. In this sense, we want to know what operation can we perform twice to turn a 1 into a -1.

- What would multiplying twice by 1 do?
- What would multiplying twice by -1 do?
- How about a **counter-clockwise rotation** of 90° twice?
- What was that last bullet point again?

This counter-clockwise rotation of 90° is equivalent to multiplying by i. Similarly, a clockwise rotation of 90° is equivalent to multiplying by $-i$.



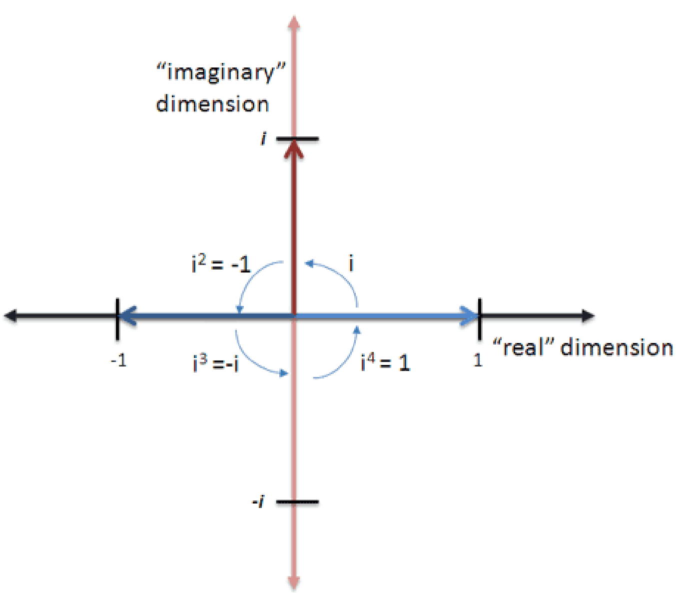


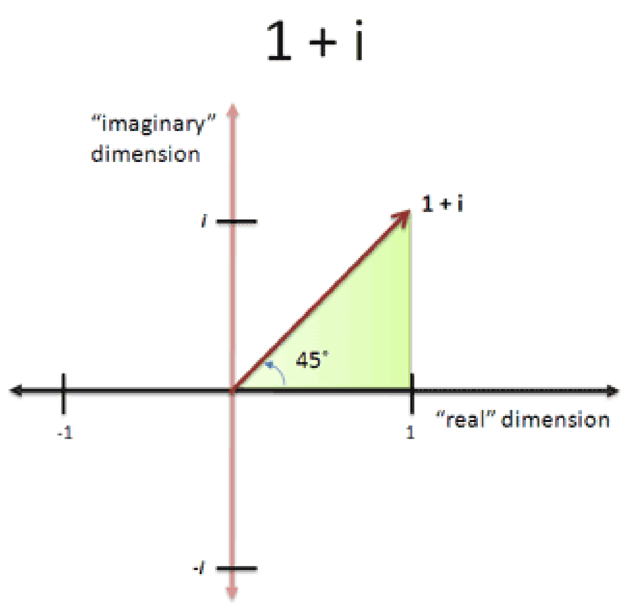
Multiplying twice by i or $-i$ rotate us 180° and bring us from 1 to -1. Thus, there are really TWO squared roots of -1: i and $-i$.

This geometric interpretation of imaginary numbers did not come until decades after Euler and Gauss began embracing them. This new coordinate plane is called the **complex coordinate plane**, with the real part on the x-axis and the imaginary part on the y-axis.

We can now visualize the repeated patterns from **Example 1** in terms of our new geometric understanding of rotation.

We already have learned how complex numbers can be entirely real, entirely imaginary, or a combination of real and imaginary. What would a complex number like $1+i$ look like on the complex plane?





Example 3:
If $u = 3+6i$ and $v = 4-2i$, evaluate and simplify the following expressions. Write all answers in $a+bi$ format. Sketch your result from (a) on the complex number plane.

(a) $u+v$ (b) $2u-3v$ (c) $-uv$

(d) $\dfrac{u}{4i}$ (e) $\dfrac{v}{u}$ (f) $\dfrac{3u}{v^2}$

We're ready for polynomials now.

Example 4:
Write a general equation of a polynomial function, $f(x)$, in reduced-factored form, whose only complex roots are $x=-2$ $(m2)$, $x = 2-3i$, and $x = 2+3i$, such that $\lim_{x \to \infty} f(x) = -\infty$.

Complex Conjugate Theorem

If $a+bi$ is a root of a polynomial function with real coefficients, then its **complex conjugate**, $a-bi$ is also a root. Therefore, complex roots occur in conjugate pairs.

Example 5:
Using factoring abilities only, find the complete factorization and all zeros for the polynomial $f(x) = 3x^5 + 24x^3 + 48x$.

Example 6:
Using the rational root theorem and your graphing calculator, find the simplified, exact value of all the roots of $P(x) = 3x^4 - 2x^3 - x^2 - 12x - 4$.

Example 7:
Find the simplified, exact values of all complex zeros for $f(x) = -13 - 46x + 36x^2 - 10x^3 + x^4$ if $x = 3 - 2i$ is a root.

Example 8:

Find the simplified, exact complex roots of $f\left(x\right)=x^5-2x^4+18x^3-36x^2+81x-162$ given that $x=-3i$ is a double root of $f\left(x\right)$.

Example 9:

Find the particular equation of a polynomial, $f\left(x\right)$, in reduced-factored form, whose only roots are $x=2\pm 6i\ (m2)$, $x=-2\pm 2\sqrt{2}$, and $x=1$, such that $f\left(0\right)=25$.