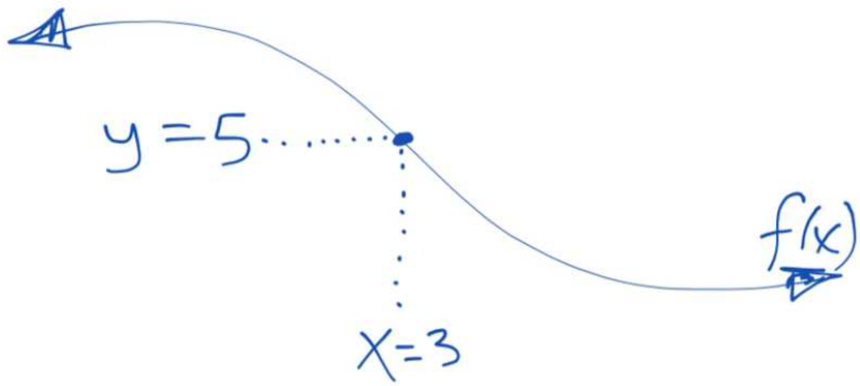
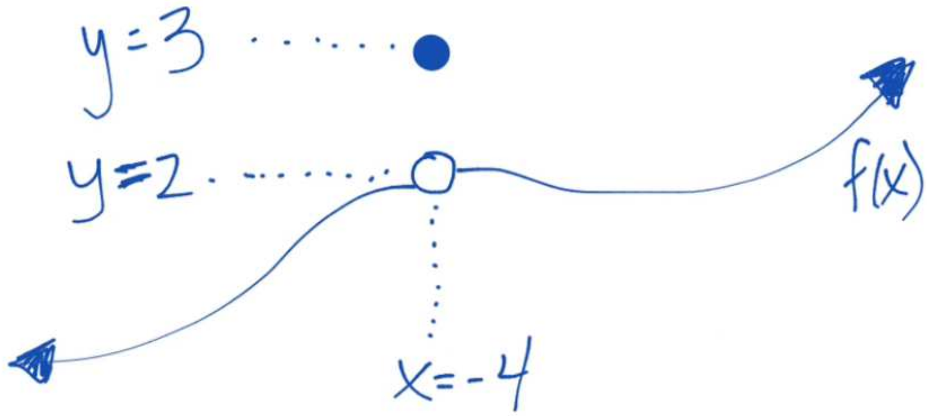
Chapter 2.2: Limits & Continuity

Sometimes when a function does not exist at a certain point, say $x = c$, we'd still like to be able to algebraically analyze what is happening at that point. Because we cannot "go" there, that is, $f(c)$ may not exist there, we can only analyze the function by approaching $x = c$ from either side. This idea is called the **limit of the function as *x* approaches *c***. We'll start be defining the limit and getting an intuitive, graphical idea of what it is and why it's needed.

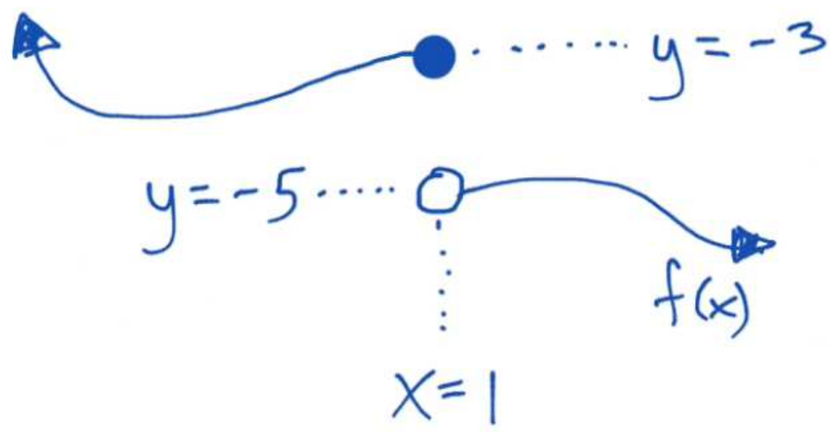
Example 1:
For the function below, analyze what is happening to the *y*-values as *x* approaches $x = 3$ from both sides of $x = 3$.



Example 2:
For the function below, analyze what is happening to the *y*-values as *x* approaches $x = -4$ from both sides of $x = -4$.



Example 3:
For the function below, analyze what is happening to the *y*-values as *x* approaches $x=1$ from both sides of $x=1$.



- We define the limit from the left side of $x=c$ to be $\lim_{x \to c^-} f(x)$
- We define the limit from the right side of $x=c$ to be $\lim_{x \to c^+} f(x)$
- The general limit, denoted as $\lim_{x \to c} f(x)$, exists only if the left- and right-sided limits exist and are equal.

Theorem (General Limit)

$$\lim_{x \to c} f(x) = L \text{ if and only if } \lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x)$$

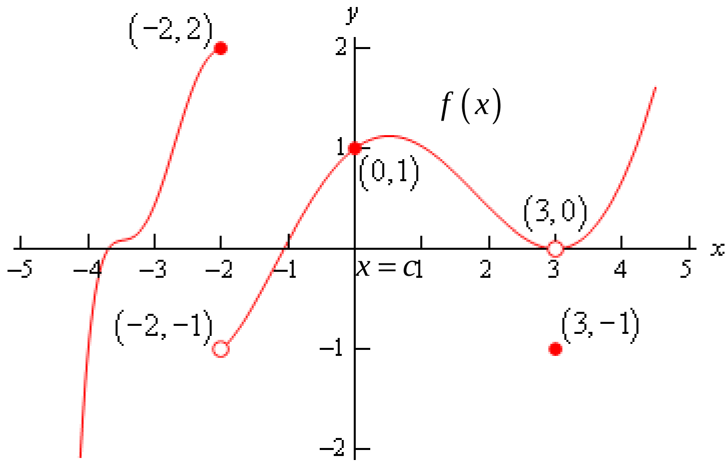
Where $y=L$ is a finite *y*-value

This essentially says that the (general) limit only exists if the two one-sided limits exist and are the same.

We can now express the different types of discontinuities in terms of the limit.

Example 4:

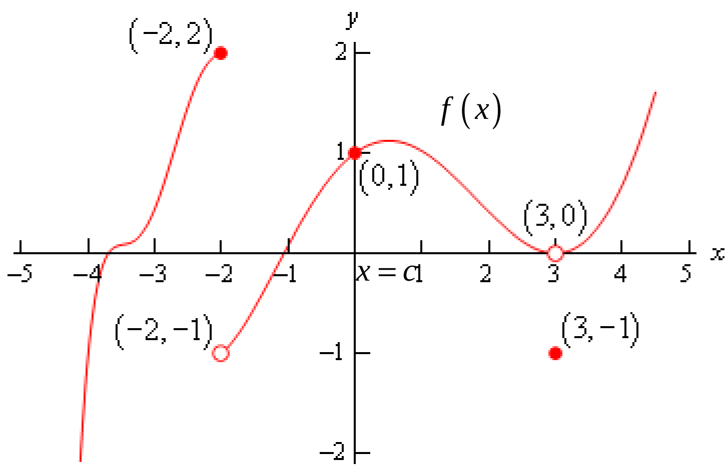
(a) When the two one-sided limits exist and are the same (that is, the limit exists), the function has either a **removable point discontinuity** (hole) or is continuous at $x = c$.



$\lim\limits_{x \to 0^-} f(x) =$ _______ and $\lim\limits_{x \to 0^+} f(x) =$ _______ so $\lim\limits_{x \to 0} f(x) =$ _______

$\lim\limits_{x \to 3^-} f(x) =$ _______ and $\lim\limits_{x \to 3^+} f(x) =$ _______ so $\lim\limits_{x \to 3} f(x) =$ _______

(b) If the two one-sided limits exist, but are different *y*-values, the function has a **non-removable jump discontinuity** at $x = c$.



$\lim\limits_{x \to -2^-} f(x) =$ _______ and $\lim\limits_{x \to -2^+} f(x) =$ _______ so $\lim\limits_{x \to -2} f(x) =$ _______

Definition

$\lim_{x\to c^-} f(x) = \infty$ or $-\infty$ AND/OR $\lim_{x\to c^+} f(x) = \infty$ or $-\infty$ if and only if there exists a **Vertical Asymptote** (VA) at $x = c$

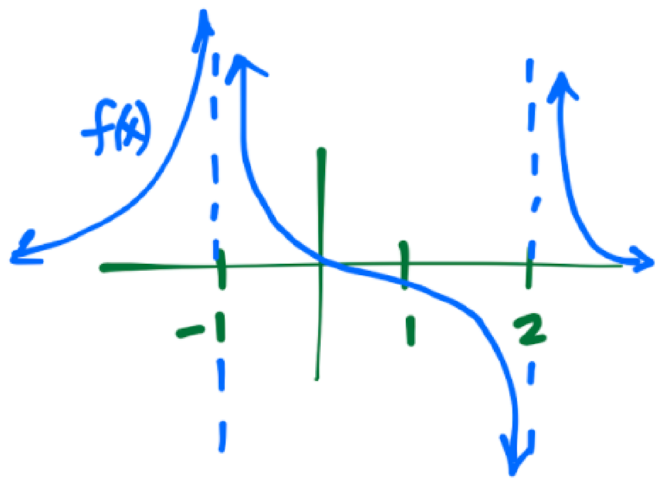
Assuming the graph of a function exists on a particular side of a vertical asymptote, there are only two options as you approach it:

(1) you can **increase without bound** (in which case the limit does not exist because it is positive infinity) or

(2) you can **decrease without bound** (in which case the limit does not exist because it is negative infinity)

Example 5a:

Given the graph at right, evaluate each of the following.



(a) $\lim_{x\to -1^-} f(x) =$ (b) $\lim_{x\to -1^+} f(x) =$

(c) $\lim_{x\to -1} f(x) =$ (d) $f(-1) =$

(e) $\lim_{x\to 2^-} f(x) =$ (f) $\lim_{x\to 2^+} f(x) =$

(g) $\lim_{x\to 2} f(x) =$ (h) $f(2) =$

Example 5b:

Sketch the graph of $f(x) = -\dfrac{4}{x+2}$ using your knowledge of transformations, then answer the following questions.

(a) $\lim_{x\to -2^-} f(x)$ (b) $\lim_{x\to -2^+} f(x)$

(c) $\lim_{x\to -2} f(x)$ (d) $f(-2)$

(e) Is $f(x)$ continuous at $x = 2$? Why or why not?

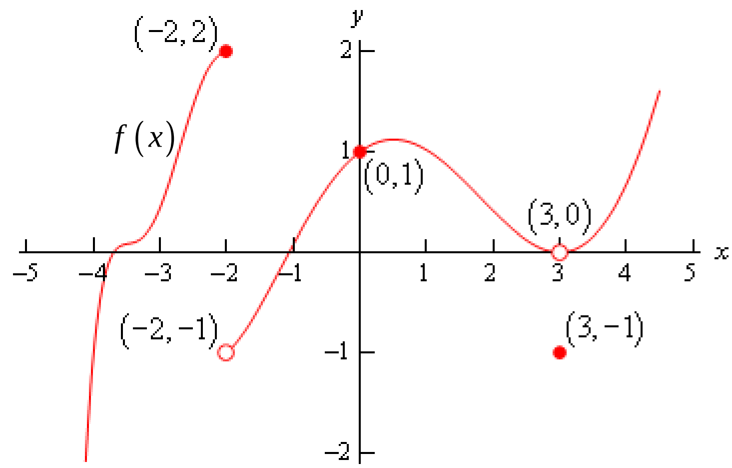
Just as we can define discontinuities in terms of the limit, we can rigidly define now, in terms of the limit, what it means for a function to be continuous at a point.

Definition of continuity at a point (3-step definition)

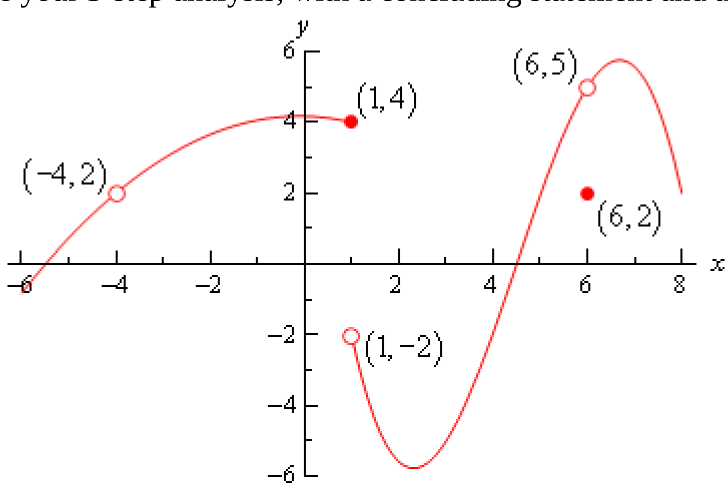
A function $f(x)$ is said to be continuous at $x=c$ if and only if.

$$\lim_{x\to c^-} f(x) = f(c) = \lim_{x\to c^+} f(x)$$

Example 6:
Using the 3-step definition of continuity at a point, determine whether the function $y=f(x)$ whose graph is given below, is continuous or not at $x=0$. Show your analysis with correct notation and give a concluding statement with justification.



Example 7:
Given the following graph of $y=f(x)$, determine if the function is continuous or not at the indicated *x*-values. Be sure to include your 3-step analysis, with a concluding statement and a justification.



(a) at $x=-4$

(b) at $x=1$

(c) at $x=6$

Example 8:
Determine if each of the following functions are continuous as the indicated value. Justify by sketching the graph .

(a) $f\left(x\right)=\begin{cases} x^2+2x-3, & x\le 1 \\ 2\sqrt{x-1}, & x>1 \end{cases}$ at $x=1$

(b) $g\left(x\right)=\begin{cases} \dfrac{6}{x}, & x<-2 \\ \dfrac{1}{2}x^2-3, & x\ge -2 \end{cases}$ at $x=-2$

When we find function values and limit values at specific, **relatively small *x*-values**, we are said to be describing the **local behavior** of the graph of the function. It is also helpful to discuss the **end behavior** of a graph (**for LARGE values of *x***).

We analyze the two ends of the graph of $f\left(x\right)$ by analyzing two one-sided limits:

$$\lim_{x\to\infty} f\left(x\right) \text{ and } \lim_{x\to-\infty} f\left(x\right)$$

Example 9:
Using the graph from Example 5, for $f\left(x\right)=-\dfrac{4}{x+2}$, analyze the end behavior by evaluating the following:

(a) $\lim\limits_{x\to\infty} f\left(x\right)$

(b) $\lim\limits_{x\to-\infty} f\left(x\right)$

(c) What graphical feature on the graph of $f\left(x\right)$ causes the graph to do this?

Definition

$\lim\limits_{x\to-\infty} f(x)=L$ AND/OR $\lim\limits_{x\to\infty} f(x)=L$, where $y=L$ is a finite *y*-value, if and only if there is a **horizontal asymptote** (HA) at $y=L$

There are two important things to clarify here about horizontal asymptotes:

1) A graph may cross its horizontal asymptote any number of times.
2) A horizontal asymptote is NOT a discontinuity (although it IS an asymptote!)

Example 10:

For each function, $f(x)$, determine (i) $\lim_{x\to-\infty} f(x)$ (ii) $\lim_{x\to\infty} f(x)$ (iii) any equations of HA's

(a) $f(x) = -3x^2 + 5x - 1$ (b) $f(x) = 4x^3 - 2$ (c) $f(x) = 2\sqrt{x-5}$

(d) $f(x) = \dfrac{3x+1}{5x^2-2}$ (e) $f(x) = \dfrac{3x-2+4x^2}{3x^2-7x+11}$ (f) $f(x) = \dfrac{4x^5+3x^2-2}{7-6x^4}$

Example 11: Putting it all together

Draw the graph of a function $f(x)$ on the interval $-5 \le x < \infty$ with the following characteristics.

$\lim_{x\to-3^-} f(x) = -2 = f(-3)$, $\lim_{x\to-3^+} f(x) = \infty$, $f(0) = 0 = \lim_{x\to0} f(x)$,

$\lim_{x\to3^-} f(x) = 4$, $\lim_{x\to3^+} f(x) = 1$, $f(3) = -1$, $\lim_{x\to\infty} f(x) = 2$

Example 12: Summary

1. If $\lim_{x\to c^-} f(x) = f(c) = \lim_{x\to c^+} f(x)$, then what can be said about $f(x)$ at $x = c$?

2. If $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$, then $\lim_{x\to c} f(x) =$

3. If $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$, then $f(x)$ has what type of discontinuity at $x = c$?

4. If $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) \neq f(c)$, then $f(x)$ has what type of discontinuity at $x = c$?

5. If $\lim_{x\to c^-} f(x) = \infty$ or $\lim_{x\to c^-} f(x) = -\infty$ or $\lim_{x\to c^+} f(x) = \infty$ or $\lim_{x\to c^+} f(x) = -\infty$, then what graphical feature does the graph of $f(x)$ have at $x = c$?

6. If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then what graphical feature does the graph of $f(x)$ have as a result?