Chapter 2.1: Algebraic Domains of Functions

Imagine going to the grocery store, selecting 5 pomegranates to purchase, get to the register, at which time the cashier tells you, “Your total is \$2.20 or \$8.50.” Not only would you be confused, but you’d jump all over the lower price (I assume). This makes no sense.

You independently chose what to buy, and you expect that the total price of your independent choices to be a single, fixed price that is entirely dependent upon your aforementioned independent choices.

This is why functions are so important: it is how we make sense of the world!

Definition of a function

A function is a relation between two variables, whereby, each **independent variable** in a **domain** maps to no more than one **dependent variable** in the **range**.

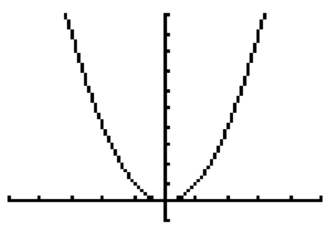
We can represent a function numerically, graphically, or analytically. For instance, the following all represent the same functional relationship.

Numeric

X	Y1	
0	0	
1	1	
2	4	
3	9	
4	16	
5	25	
6	36	

X=0

Graphic



Analytic/Algebraic

$$f(x) = x^2$$

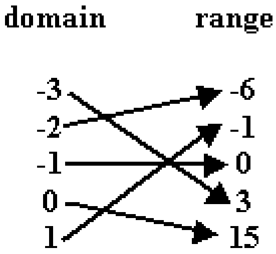
In the analytic representation above, the name of the function is “*f*,” the independent variable is “*x*,” and the dependent (*y*-value) is the entire “ $f(x)$.” The numeric representation may also be represent as $\{(0,0),(1,1),(2,4),(3,9)(4,16),(5,25),(6,36)\}$.

If you were to plot the independent variables on the *x*-axis and the dependent variables on the *y*-axis, then connected the dots, the resulting graph would pass the **vertical line test**. Repeated outputs are fine, as long as they come from a different input. Repeated inputs violate the function relationship.

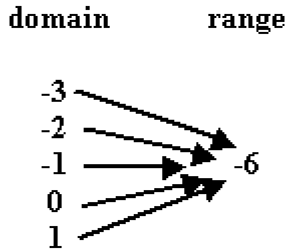
Example 1:

Determine if the following represent functions. If not, explain why.

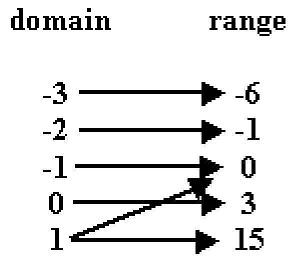
(a)



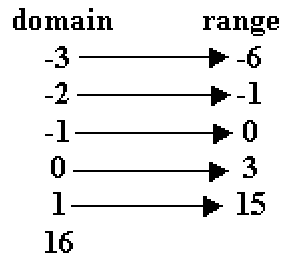
(b)



(c)



(d)



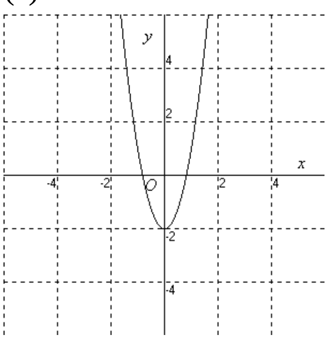
(e) {(6, 1), (3, 1), (6, –5), (2, 7), (–8, 3)}

(f) {(5, 0), (5, –2), (2, –1), (4, –1)}

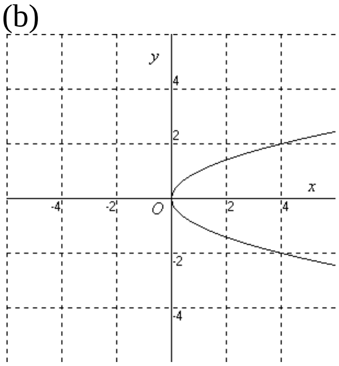
Example 2:

Determine if the following represent functions. If not, explain why.

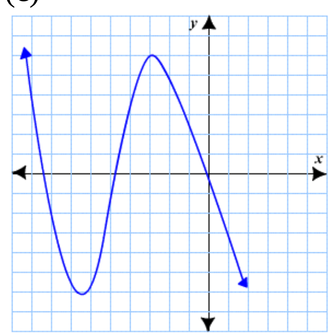
(a)



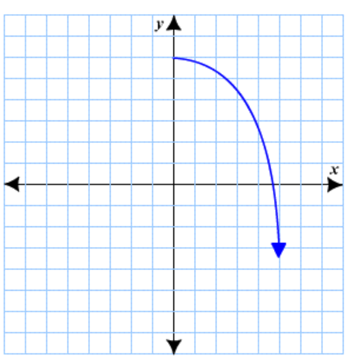
(b)



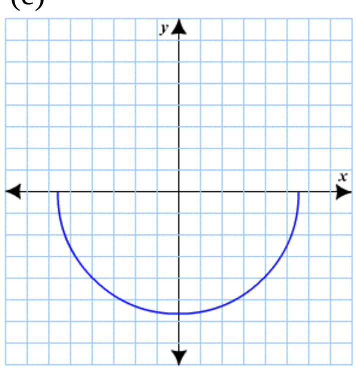
(c)



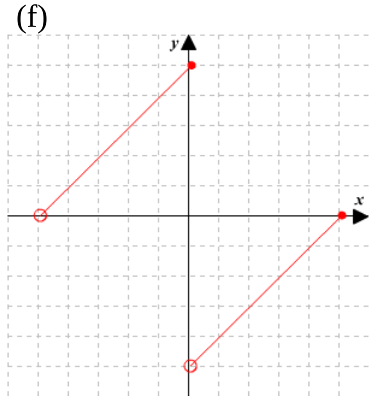
(d)



(e)



(f)



Deciding if an equation that is not explicitly solved for $f(x)$, $g(x)$, $p(x)$, etc. but rather has *x*'s and *y*'s is pretty easy. Think of all the graphing you've done so far, especially on the calculator. You've had to explicitly solve them for *y*. Equations such as $3y+5x=7$ can easily be solved for *y*. This is a function.

What about an equation like $x^2+y^2=10$? Can you solve for *y* exclusively? Do you take the positive or negative square root? The answer is BOTH, therefore, it's not a function. In general, if the powers of *y* indicates how many *y*-values are associated with each, individual *x*-value. If there is a power greater than one, the equation will represent a relation/graph that is generally NOT a function. Look for even powers of *y* or mixed powers of *y*.

Example 3:
Determine if the following represent functions. If not, explain why.
(a) $y^2+3x=6$ (b) $2x^2y+4-8y=7x$ (c) $y=x^2y+xy^{3/2}$ (d) $x=\sqrt{y}$

Once you've identified a function, it is very, very, very, very important to identify the domain of the function. We are going to be working mostly with equations of functions sans calculator, and we should get in the habit of always, always considering the domain of any function we are working with at the onset. After all, it doesn't make sense to talk about a function where it doesn't exist. You wouldn't put a metal rod into a wood chipper, would you?

To find the **<u>Real</u>** domains of functions algebraically, we primarily are looking for values of *x* that DON'T work. Unless we're dealing with piecewise functions, which are the exception to every rule, there only three things we need to watch out for.

1) Division by zero
2) Taking an even root of a negative number
3) Taking the log of a non-positive number

At this point in the course, we will only concern ourselves with the first two. The time to worry about the third one will come soon enough, believe me.

Example 4:
Find the domain of each of the following. State your answer in set or interval notation.
(a) $f(x)=\dfrac{3x}{x^2+x-6}$ (b) $g(x)=\dfrac{3x+12}{x^2-16}$ (c) $h(x)=\sqrt{2x-3}+1$ (d) $p(x)=4-7\sqrt{3-5x}$

Sometimes we have to look out for both cases in a single function.

Example 5:
Find the domain of each of the following. State your answer in set or interval notation.

(a) $y=\dfrac{\sqrt{2x-1}}{x^2-3x-10}$ (b) $m(t)=\dfrac{6t}{\sqrt{4-7t}}$ (c) $y=\dfrac{\sqrt{-x}}{x^2+5}$

(d) $y=\dfrac{15x-3}{\sqrt{3+5x^2}}$ (e) $f(x)=\dfrac{\sqrt{5x-3}}{x^2+2x-3}$ (f) $g(x)=\dfrac{\sqrt{5x-3}}{x^2+4x+3}$ (g) $r(x)=\dfrac{\sqrt{5x-3}}{x^2-4x+3}$

Example 6:
Find the domain of each of the following. State your answer in set or interval notation.

(a) $f(x)=-6x^2+\sqrt{3}x+\dfrac{2}{3}$ (b) $g(x)=4^x-5$ (c) $h(t)=\dfrac{\sqrt[3]{t-1}}{2\sqrt{3}}$

As easy as it can sometimes be, sometimes we need to roll up our sleeves and apply a little elbow grease.

Example 7:
Find the domain of each of the following. State your answer in set or interval notation.

(a) $f(x)=\dfrac{9x+7}{\sqrt{x-2}+1}$ (b) $q(x)=\dfrac{6x+11}{\sqrt{x+5}-2}$ (c) $p(x)=4\sqrt{x^2-4}-4$

(verify on your calculator)

(d) $m(x)=4\sqrt{4-x^2}-4$ (e) $s(t)=\dfrac{6t}{\sqrt{3-2t^2}}$ (f) $u(n)=\dfrac{\dfrac{1}{n}+\dfrac{1}{2}}{\dfrac{n-1}{n}}$

(verify on your calculator)

For domains of functions, division by zero and even-roots of negative numbers are the same but different. Neither are allowed, but division by zero results from a finite set of real numbers (we're trying to avoid a **single** number, ZERO), while even-roots of negatives results in an infinite interval of real numbers (we're trying to avoid a **type** of number, NEGATIVES.)

Definition

For any function $f(x)$, any value of x than results in division by zero is called a **discontinuity** of $f(x)$, so called because the graph exists to the left and to the right, and therefore, interrupts the graph of $f(x)$. These discontinuities are classified as either **Removable Point Discontinuities** (holes) or **Non-Removable Infinite Discontinuities** (Vertical Asymptotes a.k.a.VA's)

Example 8:
Find the domain of each of the following, then find the discontinuities of each of the following functions. Classify each as a hole or a VA

(a) $f(x)=\dfrac{2x^2-5x+2}{x-2}$ (b) $g(x)=\dfrac{2x^2+3x-2}{x-2}$ (c) $T(x)=\dfrac{x^2-4x-12}{x^2-4}$

An x-value that yields a zero in the denominator means that that x-value is a) not in the domain and is b) a discontinuity, but the value that the same x-value yields in the numerator will reveal which type of discontinuity.

$$\frac{0}{0} \rightarrow \mathrm{H}\frac{0}{0}\mathrm{LE} \qquad \frac{\neq 0}{0} \rightarrow VA \qquad \text{By the way, } \frac{0}{\neq 0} = 0$$

There is a third type of discontinuity that doesn't show up too often (outside a math classroom, that is). They are called **Non-Removable Jump Discontinuities**. As the name implies, they are both non-removable and discontinuous, but they are also JUMPS—the y-values "jump" from one to another at the x-value. These usually manifest themselves by way of piece-wise functions, but they can also sneak into existence by way of the following:

Example 9:

Graph the function $f(x) = \dfrac{|5x-15|}{x-3}$, by decomposing it into a piecewise function. State the domain and classify any discontinuities. Verify by graphing on the calculator.

Example 10:

Graph the following piecewise functions. State the domain and classify any discontinuities. Verify by graphing on the calculator.

(a) $f(x) = \begin{cases} x-1, & x \leq 0 \\ x^2, & x > 0 \end{cases}$

(b) $f(x) = \begin{cases} x-1, & x < 0 \\ x^2, & x > 0 \end{cases}$

(c) $f\left(x\right)=\begin{cases} x-1, & x<0 \\ 1, & x=0 \\ x^2, & x>0 \end{cases}$

(d) $f\left(x\right)=\begin{cases} x-1, & x\le 0 \\ x^2-1, & x>0 \end{cases}$