Chapter 1.1: Exponents

If you're sitting in this class, you've had Algebra I, Geometry, and Algebra II. Almost everything you learned in those classes, you're going to need to remember in order to master Precalculus. It's a good thing you remember all of it still.

Precalculus is basically two courses in one: Algebraic Analysis, sometimes called Analytic Geometry, and Trigonometry.

In the first semester, we will learn Analytic Geometry. This can be described as a souped-up version of Algebra II. We will revisit polynomial functions, rational functions, radical functions, exponential functions, and finally, logarithmic functions. We will be studying these functions in much greater depth and the problems we work will be much more challenging, very worthy of your time.

In the second semester, we will learn something that is entirely new to most people: Trigonometry. Picking up where Geometry leaves off, we'll revisit what SOH CAH TOA really means, memorize something called the Unit Circle, and try our hands at trigonometric proofs. It's sure to be a lot of fun, a lot of memorizing, and a lot of fun.

Be warned, though. This course is usually when most PreAP math students "hit the wall." It's important you work through that wall, climb over that wall, or simply walk around it. Stopping at the wall not only means you don't "want it" enough, it all but guarantees you will not be successful in this course. Although this course is challenging, it is entirely possible to survive it. Your mastery and final grade will be directly proportional to the quality and quantity of your efforts in this class.

So, what are the absolute essential prerequisite skills for mastering Precalculus?

<u>I. Understanding what an exponent is.</u>

Just like multiplication is abbreviated addition, $2+2+2=3(2)$, exponentiation is abbreviated multiplication, $2\cdot 2\cdot 2=2^3$. In general, an expression of the form

$$b^n$$

is exponential, where b is the nonzero real base, and n is the exponent

If n is a positive whole number (also called a "counting" or "natural number), that is $n\in\mathbb{N}$, then n tells you how many times to multiply the base, b, times itself, that is, the number of <u>**FACTORS**</u> of the base there are.

Say it: FACTORS
MULTIPLICATION
EXPONENTS
FACTORS
WORCESTERSHIRE

Example 1:

Express each expression in the form b^n such that $n \in \mathbb{N}$

(a) $5\cdot5\cdot5\cdot5\cdot5\cdot5$ (b) $\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)$ (c) $-3.1^4\left[(-3.1)^3\right]^2\cdot(-3.1)^5$

(d) $(-x)^4\left[(-x)^3\right]^2\cdot(-x)^5$ (e) $\left(2^3\right)^4\cdot\left(4^3\right)^2$ (f) $\dfrac{49\cdot9\cdot7^5}{7^3\cdot3^2}$

Once again, EXPONENTIATION IS ABBREVIATED MULTIPLICATION!

We can summarize some of the underpinning ideas we intuitively utilized above to form some basic rules of exponents. These rules will eventually allow us to broadly apply to expressions with not-so-friendly-looking bases and exponents.

Property	**Example**
1. $u^m u^n = u^{m+n}$	$3^4\cdot3^5 = 3^{4+5} = 3^9$
2. $\dfrac{u^m}{u^n} = u^{m-n}$	$\dfrac{5^8}{5^6} = 5^{8-6} = 5^2$
3. $u^0 = 1$	$10^0 = 1$
4. $\left(u^m\right)^n = u^{mn}$	$\left(8^7\right)^6 = 8^{42}$
5. $(uv)^m = u^m v^m$	$\left(3x^2y\right)^3 = 3^3x^6y^3$
6. $\left(\dfrac{u}{v}\right)^m = \dfrac{u^m}{v^m}$	$\left(\dfrac{2}{3}\right)^4 = \dfrac{2^4}{3^4}$

***Property 3: Why is anything (other than zero) to the zeroth power equal to one? Which property can be used to prove this? Why can the base not be zero?**

<u>II. Understanding Negative Exponents</u>

A NEGATIVE exponent generally implies a FRACTION (and vice-versa)

If $b \in \mathbb{R}$, $b \neq 0$ and $n \in \mathbb{R}^+$, then

$$b^{-n} = \frac{1}{b^n}$$

Example 2:
Simplify the following exponential expressions, if possible.

(a) $\dfrac{2}{x^{-4}}$ (b) $3x^{-2}$ (c) $(3x)^{-2}$ (d) $\dfrac{4y^3}{2\left(y^2\right)^4}$

III. Understanding Fractional (decimal/Rational) Exponents
A FRACTIONAL exponent or a decimal exponent implies a RADICAL (and versa-vice)

If $b \in \mathbb{R},\ b \neq 0$ and $n, m \in \mathbb{R}^+,\ n \neq 0$, then

$$b^{m/n} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$$

Example 3:
Simplify the following exponential expressions, if possible.

(a) $3\sqrt[4]{x^2}$ (b) $\dfrac{4}{5\sqrt{2^5}}$ (c) $\sqrt{x} \cdot \sqrt[3]{x}$ (d) $\dfrac{\left(2\sqrt[5]{x^2}\right)^2}{3\sqrt{x^5}}$ (e) $4^{3/2}$

Summary of the Properties of Exponents
Let *u* and *v* be real numbers, variables, or algebraic expressions, and let *m* and *n* be integers. Assume no bases or denominators equal 0.

Property	Example
1. $u^m u^n = u^{m+n}$	$3^4 \cdot 3^5 = 3^{4+5} = 3^9$
2. $\dfrac{u^m}{u^n} = u^{m-n}$	$\dfrac{5^8}{5^6} = 5^{8-6} = 5^2$
3. $u^0 = 1$	$10^0 = 1$
4. $\left(u^m\right)^n = u^{mn}$	$\left(8^7\right)^6 = 8^{42}$
5. $(uv)^m = u^m v^m$	$\left(3x^2y\right)^3 = 3^3 x^6 y^3$
6. $\left(\dfrac{u}{v}\right)^m = \dfrac{u^m}{v^m}$	$\left(\dfrac{2}{3}\right)^4 = \dfrac{2^4}{3^4}$
7. $u^{-n} = \dfrac{1}{u^n}$	$\dfrac{4^3}{4^7} = 4^{3-7} = 4^{-4} = \dfrac{1}{4^4}$
8. $\sqrt[m]{u^n} = u^{n/m}$	$\sqrt[3]{5^4} = 5^{4/3} = \left(\sqrt[3]{5}\right)^4$
9. $\dfrac{v}{\sqrt[m]{u^n}} = v \cdot u^{-n/m}$	$\dfrac{5}{\sqrt[3]{6^2}} = 5 \cdot 6^{-2/3}$

So WHY do we need to have all these rules? Why express fractions as negative exponents? Why express radicals as fractional exponents?

When we are consolidating factors, that is, simplifying expressions with multiple factors and exponents, the exponential properties allow us to use simple addition & multiplication to get the job done. In general, fractions and radicals are easier to work with when evaluating numbers or expressing final answers, while negative and fractional exponents are preferred with simplifying "ugly" expressions with multiple fractions and exponents.

IV. Manipulating, consolidating, and simplifying expressions with exponents

Keep this in mind when simplifying expressions:

- Rewrite radicals as rational exponents.
- Don't work with decimals. It's easier to work with rational numbers.
- There is usually more than one correct way to work the problem. Inch by inch it's a cinch.
- As you work, you need to get used to working straight down so that each line should be equivalent to the previous one, and someone should be able to follow from line to line and determine what was done. Don't skip steps.
- "Simplifying" generally means: making expressions as "small" as possible, combining like terms, and dividing out common factors.
- We do not want **numeric radicals** in a denominator, so we rationalize them by multiplying by a clever form of one.
- Look to simplify all radical expressions, being sure to look for factors of radicands that are perfect squares.

Example 4:

Simplify the following expressions without a calculator and without sweating.

(a) $\left(2ab^3\right)^2\left(5\sqrt{ab^5}\right)$ (b) $\sqrt{\dfrac{(2u)^5 v^{-2}}{3u^{-1}v^3}}$ (c) $\left(\dfrac{2x^3y^{-2}x^{\frac{1}{2}}}{\sqrt[4]{y^3x^{\frac{1}{2}}}}\right)^{-3}\cdot\left(\sqrt[3]{4xy^{-6}}\right)$

Now that we know what is CORRECT, let's compare it next to what is INCORRECT.

CORRECT <u>MULTIPLICATION</u> PROPERTY	COMMON <u>ADDITION</u> **ERROR**
$(a \cdot b)^2 = a^2 \cdot b^2$	$(a+b)^2 \neq a^2 + b^2$
$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$	$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$
$\sqrt{a^2 \cdot b^2} = a \cdot b$	$\sqrt{a^2 + b^2} \neq a + b$
$\frac{1}{a \cdot b} = \frac{1}{a} \cdot \frac{1}{b}$	$\frac{1}{a+b} \neq \frac{1}{a} + \frac{1}{b}$
$\frac{a \cdot b}{a} = b$	$\frac{a+b}{a} \neq b$
$(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$	$(a+b)^{-1} \neq a^{-1} + b^{-1}$

A **counterexample** is an example that disproves a statement, claim, or equation. While an equation may be true for many, many, values (or just one), we say it is not true if there is at least one counterexample to disprove it.

Example 5:

Provide a counterexample for the claim that $(a+b)^2 = a^2 + b^2$.