Name KEY Date Angle of Incidence

AP Calculus TEST 4.1-5.1, No Calculator

Section 1: Multiple Choice—You know what to do

C 1. What is the average value of $f(x)=\sqrt{x}$ on the closed interval $[0,3]$?

(A) $\frac{\sqrt{3}}{3}$ (B) $\frac{3\sqrt{3}}{2}$ (C) $\frac{2\sqrt{3}}{3}$ (D) $2\sqrt{3}$ (E) $\sqrt{3}$

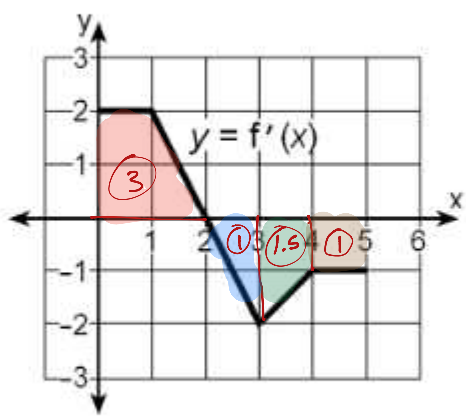
$$\text{Avg value} = \frac{\int_0^3 \sqrt{x}\,dx}{3-0}$$
$$= \frac{1}{3}\int_0^3 x^{1/2}\,dx$$
$$= \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)x^{3/2}\Big|_0^3$$
$$= \frac{2}{9}\left[3^{3/2}-0^{3/2}\right]$$
$$= \frac{2}{9}\cdot 3\sqrt{3} = \frac{2\sqrt{3}}{3}$$

D 2. If $F(x)=\int_0^x \sqrt{t^2-9}\,dt$, then $F'(5)=$

(A) $\frac{5}{4}$ (B) -4 (C) $-\frac{5}{4}$ (D) 4 (E) 16

$$F'(x)=\sqrt{x^2-9}\,(1)-\sqrt{0^2-9}\,(0)$$
$$F'(x)=\sqrt{x^2-9} \quad (\text{FTOC part II})$$
$$F'(5)=\sqrt{5^2-9}$$
$$=\sqrt{25-9}$$
$$=\sqrt{16}$$
$$=4$$

C 3. The graph below sows the graph of f', the derivative of function f.



If $f(5)=2$, then $f(0)=$

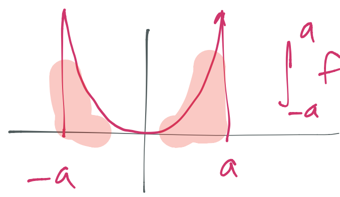
(A) $\frac{3}{2}$ (B) $-\frac{1}{2}$ (C) $\frac{5}{2}$ (D) 2 (E) $\frac{1}{2}$

$$f(0)=f(5)+\int_5^0 f'(x)\,dx$$ Right to Left
$$=2+[1+1.5+1-3]$$
$$=2+[.5]=2.5=\frac{5}{2}$$

B 4. If the function f is integrable on $[-a,a]$ and $\underbrace{f(-x)=f(x)}_{\text{even function/y-axis symmetry}}$, then which of the following must be true?

(A) $\int_{-a}^{a} f(x)\,dx = \frac{1}{2}\int_{0}^{a} f(x)\,dx$ (B) $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ (C) $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx$

(D) $\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(x)\,dx$ (E) $\int_{-a}^{a} f(x)\,dx = \frac{1}{2}\int_{0}^{a} f(x)\,dx$



$\int_{-a}^{a} f(x)\,dx$

$= 2\int_{0}^{a} f(x)\,dx$

A 5. $\int x\sqrt{x-1}\;dx =$

(A) $\frac{2}{5}(x-1)^{5/2}+\frac{2}{3}(x-1)^{3/2}+C$ (B) $\frac{1}{2}(x-1)^{4}+C$ (C) $\frac{5}{2}(x-1)^{5/2}+\frac{3}{2}(x-1)^{3/2}+C$

(D) $\frac{1}{3}x^{2}(x-1)^{3/2}+C$ (E) $\frac{2}{3}(x^{2}-x)^{3/2}+C$

u-sub

$u = x-1$
$x = u+1$
$dx = du$

$\int (u+1)\,u^{1/2}\,du$

$\int \left(u^{3/2}+u^{1/2}\right)du$

$\frac{2}{5}u^{5/2}+\frac{2}{3}u^{3/2}+C$

$\frac{2}{5}(x-1)^{5/2}+\frac{2}{3}(x-1)^{3/2}+C$

B 6. $\frac{d}{dx}\left[\int_{2x}^{x^2} \cos^2 t\;dt\right] =$ FTOC (part II)

(A) $2x\left[\cos^2(x^2)-\cos^2(2x)\right]$ (B) $2\left[x\cos^2(x^2)-\cos^2(2x)\right]$ (C) $2x^2\cos^2(x^2)$

(D) $\cos^2(x^2)-\cos^2(2x)$ (E) $\cos^2(x^2)$

$= \cos^2(x^2)\cdot 2x - \cos^2(2x)\cdot 2$

$= 2\left[x\cos^2(x^2)-\cos^2(2x)\right]$

D 7. $\int \left(x^2+2x+1\right)^{10} dx =$

(A) $\frac{(x+1)^{19}}{19}+C$ (B) $\frac{1}{11}\left(\frac{x^3}{3}+x^2+x\right)^{11}+C$ (C) $\frac{\left(x^2+2x+1\right)^{11}}{11}+C$

(D) $\frac{(x+1)^{21}}{21}+C$ (E) $\frac{(x+1)^{13}}{13}+C$

$\int \left[(x+1)^2\right]^{10} dx$

$\int (x+1)^{20} dx$

$\frac{1}{21}(x+1)^{21}+C$

B 8. $\int \left(x^2+1\right)\sqrt[5]{\left(x^3+3x-7\right)^3}\, dx =$

(A) $\frac{5}{16}\left(x^2+1\right)\left(x^3+3x-7\right)^{8/5}+C$ (B) $\frac{5}{24}\left(x^3+3x-7\right)^{8/5}+C$ (C) $\frac{1}{3}\left(x^3+3x-7\right)^{-2/5}+C$

(D) $\frac{5}{8}\left(x^3+3x-7\right)^{8/5}+C$ (E) $\frac{8}{15}\left(x^3+3x-7\right)^{8/5}+C$

$\int (x^2+1)(x^3+3x-7)^{3/5} dx$

$3x^2+3=3(x^2+1)$

off by 3

$\left(\frac{1}{3}\right)\left(\frac{5}{8}\right)(x^3+3x-7)^{8/5}+C$

correction Rule

$\frac{5}{24}(x^3+3x-7)^{8/5}+C$

B 9. If $\int_9^{19} f(x)\,dx = a$, then $\int_5^{10} f(2x-1)\,dx =$

(A) $10a$ (B) $\frac{a}{2}$ (C) $10(2a-1)$ (D) $2a-1$ (E) $2a$

u-sub

$u=2x-1$

$du=2dx$

$dx=\frac{1}{2}du$

when $x=5$, $u=2(5)-1=9$

when $x=10$, $u=2(10)-1=19$

$\int_5^{10} f(2x-1)dx = \int_9^{19} f(u)\cdot\frac{1}{2}du = \frac{1}{2}\int_9^{19} f(u)du = \frac{1}{2}a = \frac{a}{2}$

A 10. Let function g be defined by $g(x)=\begin{cases}\sin 2x, & x\le 0\\ \cos\frac{x}{2}, & x>0\end{cases}$. Evaluate $\int_{-\pi}^{\pi} g(x)\,dx$.

(A) 2 (B) 4 (C) 0 (D) -2 (E) -1

$\int_{-\pi}^{\pi} g(x)dx = \int_{-\pi}^{0}\sin 2x\,dx + \int_0^{\pi}\cos\frac{x}{2}dx$

$-\frac{1}{2}\cos 2x\Big|_{-\pi}^{0} + 2\sin\frac{x}{2}\Big|_0^{\pi}$

$-\frac{1}{2}[\cos 0-\cos(-2\pi)] + 2[\sin\frac{\pi}{2}-\sin 0]$

$-\frac{1}{2}[1-1]+2[1-0]$

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Part II: Free Response—Do and show all work in the space provided. Have fun!

11. Consider the differential equation $\dfrac{dy}{dx} = \dfrac{y^2}{x-1}$.

(a) Find $\dfrac{d^2y}{dx^2}$ in terms of x and y. For $x \ge 1$ and $y \ge 1$, is $\dfrac{d^2y}{dx^2}$ positive or negative?

$$\frac{dy}{dx} = \frac{y^2}{x-1}$$

$$\frac{d}{dx}: \frac{d^2y}{dx^2} = \frac{(x-1)(2y\frac{dy}{dx}) - (y^2)(1)}{(x-1)^2} \quad √1$$

$$= \frac{(x-1)(2y)\left(\frac{y^2}{x-1}\right) - y^2}{(x-1)^2} \quad √2$$

$$\frac{d^2y}{dx^2} = \frac{2y^3 - y^2}{(x-1)^2}$$

$$= \frac{y^2(2y-1)}{(x-1)^2}$$

for $x \ge 1$ & $y \ge 1$

$\boxed{\frac{d^2y}{dx^2} \text{ is POSITIVE}}$ √3

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the tangent line to the graph of $y = f(x)$ at $x = 2$.

pt: $(2,3)$

$$\text{slope} = m = \left.\frac{dy}{dx}\right|_{(2,3)} = \frac{3^2}{2-1} = 9$$

eq: $y = 3 + 9(x-2)$ √4

(c) Use your equation from part (a) to approximate $f(2.1)$. Is your approximation an over- or an under-approximation of $f(2.1)$. Justify.

$$f(2.1) \approx y(2.1) = 3 + 9(2.1-2) \quad √5$$
$$= 3 + 9(.1)$$
$$= 3.9$$

from part (a), $\frac{d^2y}{dx^2} > 0 \ \forall x, y \ge 1$ √6

so $y(2.1)$ underapproximates $f(2.1)$

(d) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

$$\frac{dy}{dx} = \frac{y^2}{x-1}$$

$$y^{-2}\,dy = \frac{1}{x-1}\,dx \quad √7$$

$$\underbrace{-y^{-1} = \ln|x-1|}_{√8} + C \quad √9$$

$$-\frac{1}{y} = \ln|x-1| + C$$

$$\frac{1}{y} = \frac{-\ln|x-1| + C}{1}$$

$$y = \frac{-1}{\ln|x-1| + C}$$

$$\text{at } (2,3):\ 3 = \frac{-1}{\ln|2-1| + C} \quad √10$$

$$3 = \frac{-1}{C}$$

$$C = -\tfrac{1}{3}$$

$$\text{So, } y = \frac{-1}{\ln|x-1| - \frac{1}{3}} \quad \text{or } y = \frac{-3}{3\ln|x-1| - 1}$$

√11

$$\text{or } y = \frac{3}{1 - 3\ln|x-1|}$$