Name KEY Date Undiscovered Element

AP Calculus AB (Take it to the HOUSE!)
SHOW ALL WORK ON THIS PAGE. NO WORK, NO CREDIT!

TEST: 3.1 to 3.8, **Calculator OK (SHOW CALCULATOR SET-UPS)**

N 1.

If $f(x) = k^2 \sin x + ke^x$, where k is a nonzero constant, has a critical point at $x = 0$, which of the following statements must be true.

$f'(0) = 0$

K) f has a relative minimum at $x = k$
L) f has a relative maximum at $x = k$
M) f has a relative minimum at $x = 0$
N) f has a relative maximum at $x = 0$
P) f has a point of inflection at $x = 0$

$f'(x) = k^2\cos x + ke^x$
$f'(0) = k^2\cos 0 + ke^0 = 0$
$= k^2 + k = 0$
$k(k+1) = 0$
$k = 0$, $\boxed{k=-1}$
k is nonzero

so $f'(x) = (-1)^2\cos x + (-1)e^x$
$f'(x) = \cos x - e^x$
$f''(x) = -\sin x - e^x$
$f''(0) = -\sin 0 - e^0$
$f''(0) = -1 < 0$

since $f''(0) < 0$ and $f'(0) = 0$, f has a local max at $x = 0$ (By 2nd Deriv Test).

O 2.

Let f be a function such that $f(4) < 0$ and $f'(4) = 0$. If $g(x) = e^{-f(x)}$ and g has a relative minimum at $x = 4$, what conclusion can be made about $f''(4)$?

$g(4) = e^{-f(4)} > 0$, $\rightarrow g'(4) = 0$, so $g''(4) > 0$ (g is cc up) $x=4$

A) $0 < f''(4) < 1$ E) $f''(4) = 0$ I) $f''(4) > 0$ O) $f''(4) < 0$ U) no conclusion

$g(x) = e^{-f(x)}$
$\ln g(x) = \ln e^{-f(x)}$
$\ln g(x) = -f(x)$
so, $f(x) = -\ln[g(x)]$

$f'(x) = \dfrac{-g'(x)}{g(x)}$
$f''(x) = \dfrac{[g(x)][-g''(x)] - [-g'(x)][g'(x)]}{[g(x)]^2}$
$f''(4) = \dfrac{-g(4)g''(4) + [g'(4)]^2}{[g(4)]^2}$

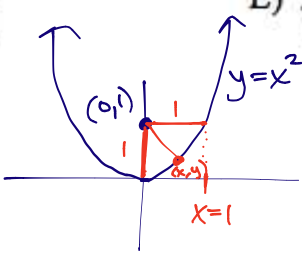
so, $g(4) = $ pos
$g'(4) = 0$
$g''(4) = $ pos

$f''(4) = \dfrac{-(+)(+) + 0^2}{(+)^2}$
$= \dfrac{-}{+}$
$= -$ (neg), so $f''(4) < 0$

I 3.

Find the shortest distance from the graph $y = x^2$ to the point (0, 1).

A) $\frac{1}{2}$ E) $\sqrt{\frac{1}{2}}$ I) $\frac{\sqrt{3}}{2}$ O) 1 U) 2



$D = \sqrt{(x-0)^2 + (y-1)^2}$
$D = \sqrt{x^2 + (y-1)^2}$
$D = \sqrt{y + (y-1)^2}$
$D' = \frac{1}{2}(y + (y-1)^2)^{-1/2} \cdot (1 + 2(y-1))$
$D' = \dfrac{2y-1}{2\sqrt{y + (y-1)^2}}$

$D' = 0$
$2y - 1 = 0$
$\boxed{y = \frac{1}{2}}$
$D(\frac{1}{2}) = \sqrt{\frac{1}{2} + (\frac{1}{2} - 1)^2}$
$= \sqrt{\frac{1}{2} + \frac{1}{4}}$
$= \sqrt{\frac{3}{4}}$
$= \frac{\sqrt{3}}{2} < 1$

so, shortest distance is $\frac{\sqrt{3}}{2}$ at $(\frac{1}{2}, \frac{1}{4})$

B 4.

Find the difference between the absolute maximum and absolute minimum value of $f(x)=\sqrt{2}\cos x-\sin^2 x$ on $[0, 2\pi]$.

$f'(x) = -\sqrt{2}\sin x - 2\sin x\cos x = 0$

$-\sin x(\sqrt{2}+2\cos x)=0$

$\sin x = 0$ or $\cos x = -\frac{\sqrt{2}}{2}$

$x=0,\pi,2\pi$ $\quad x=\frac{3\pi}{4},\frac{5\pi}{4}$

A. $2\sqrt{2}$ B. $\frac{3+2\sqrt{2}}{2}$ C. $\sqrt{2}-\frac{3}{2}$ D. $\frac{\sqrt{2}}{2}$ E. $\frac{3\sqrt{2}}{2}$

$f(0)=\sqrt{2}$

$f(\pi)=-\sqrt{2}$

$f(2\pi)=\sqrt{2} \leftarrow$ MAX

$f(\frac{3\pi}{4})=-1-\frac{1}{2}=-\frac{3}{2}$

$f(\frac{5\pi}{4})=-1-\frac{1}{2}=-\frac{3}{2} \leftarrow$ MIN

$\sqrt{2}-(-\frac{3}{2})$

$\sqrt{2}+\frac{3}{2}$

$\frac{2\sqrt{2}}{2}+\frac{3}{2}$

$\frac{2\sqrt{2}+3}{2}$

$\boxed{\frac{3+2\sqrt{2}}{2}}$

D 5.

(Calculator Active) Find the maximum slope of $f(x)=\ln(2x+1)+\cos x$ for $0\le x\le 2\pi$.

A. 0.925 B. 1.653 C. 1.193 D. 2 E. 4.675

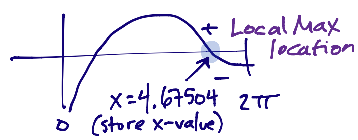
Slope function $= f'(x) = \frac{2}{2x+1} - \sin x$

$f'(x) = 2(2x+1)^{-1} - \sin x$

find critical values of f' $\rightarrow$ $f''(x) = -2(2x+1)^{-2}(2) - \cos x$

$f''(x) = \frac{-4}{(2x+1)^2} - \cos x = 0$

graph



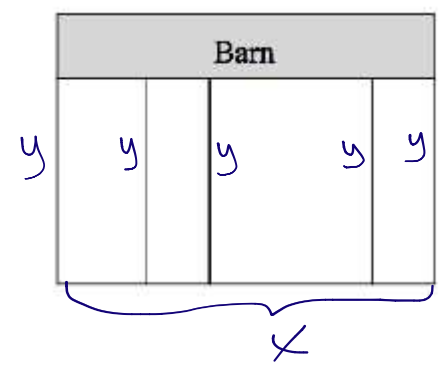
So, $f'(4.675\ldots) = 1.192537829$ (C.V.)

end points $\begin{cases} f'(0)=2 \\ f'(2\pi)=\frac{2}{4\pi+1}<0 \end{cases}$

So, max slope is 2 at $x=0$

D 6.

A farmer needs to enclose a rectangular area and create 4 pens, not necessarily the same size as shown by the figure to the right. He has 800 feet of fencing and does not need fencing against the barn. What is the largest area he can enclose?



A. 8,000 ft^2 B. 12,000 ft^2 C. 16,000 ft^2
D. 32,000 ft^2 E. 64,000 ft^2

$A = xy$

$A=(800-5y)y$

$A=800y-5y^2$

$A'=800-10y=0$

$\boxed{y=80}$

parabola opening down so $y=80$ maximizes Area

Fencing $= F = 800$

$F = 5y + x$

$5y + x = 800$

$\boxed{x = 800-5y}$

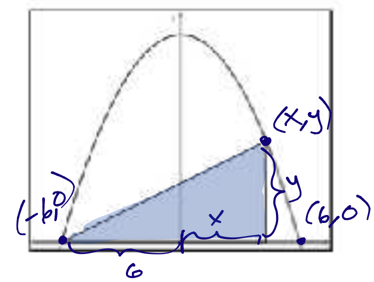
So, max area is $(800-5(80))(80) = (800-400)(80)$

$= 400(80)$

$= \boxed{32{,}000\ ft^2}$

C 7.

A right triangle is inscribed under the curve $y = 36 - x^2$ as shown by the figure to the right. What is the largest area triangle that is possible?



A. 32 B. 64 C. 128
D. 144 E. 256

Area $= \frac{1}{2}bh$

$A = \frac{1}{2}(x+6)y$

$A = \frac{1}{2}(x+6)(36-x^2)$

$A' = \frac{1}{2}[(1)(36-x^2)+(x+6)(-2x)]$

$A' = \frac{1}{2}[(6-x)(x+6) + (x+6)(-2x)]$

$A' = \frac{1}{2}(x+6)[6-x-2x]$

$A' = \frac{1}{2}(x+6)(6-3x) = 0$

$\boxed{x=-6}$ or $\boxed{x=2}$

min
Collapses triangle

So, largest area is when $x=2$,

$A = \frac{1}{2}(2+6)(36-2^2)$

$A = \frac{1}{2}(8)(32)$

$A = 8(16)$

$A = 128$ units^2

Q 8.

If $f(x) = \dfrac{2x^2 - 6x}{\sqrt{x}}$, find the equation of the tangent line to f at $x = 9$.

M) $y = 12x - 72$ N) $y = 72x - 612$ P) $-8x + 108$ Q) $y = 8x - 36$ R) $8x - 9$

$f(x) = \dfrac{2x^2}{x^{1/2}} - \dfrac{6x}{x^{1/2}}$

$f(x) = 2x^{3/2} - 6x^{1/2}$

$f'(x) = 3x^{1/2} - 3x^{-1/2}$

$f'(x) = 3\sqrt{x} - \dfrac{3}{\sqrt{x}}$

$f(9) = \dfrac{2(81)-54}{3}$

$= \dfrac{162-54}{3}$

$= \dfrac{108}{3}$

$f(9) = 36$

pt: $(9,36)$

$f'(9) = 3\sqrt{9} - \dfrac{3}{\sqrt{9}}$

$= 9 - 1$

$f'(9) = 8$

$m = 8$

$y = 36 + 8(x-9)$

$y = 36 + 8x - 72$

$\boxed{y = 8x - 36}$

H 9.

For the function f, $f(5) = -3$ and $f'(x) = 2x - 4$. What is the approximation for $f(5.2)$ found by using the tangent line to the graph of f at $x = 5$?

H) -1.8 J) -1.72 K) -1 L) 4.2 M) 6.4

pt: $(5,-3)$

$m = f'(5) = 2(5) - 4 = 6$

$\mathcal{L}(x) = -3 + 6(x-5)$

$f(5.2) \approx \mathcal{L}(5.2) = -3 + 6(5.2-5)$

$= -3 + 6(0.2)$

$= -3 + 1.2$

$= \boxed{-1.8}$

A 10.

If $f(x)=(1-2x)^2$, find the difference between $f(0.9)$ and the linear approximation to $f(0.9)$ at $x=1$.

A. 0.04 B. 0.6 C. 0.16 D. 0.64 E. 0.8

$f(0.9)=(1-2(0.9))^2$
$=(1-1.8)^2$
$=(0.8)^2$
$=0.64$

$f'(x)=2(1-2x)'(-2)$
$f'(x)=-4(1-2x)$
$f'(1)=-4(-1)$
$=\boxed{4=m}$

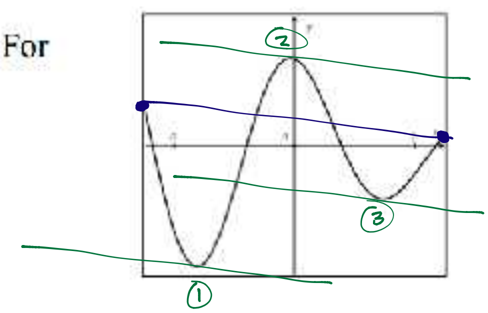
$f(1)=(1-2)^2=1$
pt: $(1,1)$
$\mathcal{L}(x)=1+4(x-1)$
$f(0.9)\approx\mathcal{L}(0.9)=1+4(0.9-1)$
$=1+4(-0.1)$
$=0.6$

so $f(0.9)-\mathcal{L}(0.9)$
$=0.64-0.6$
$=\boxed{0.04}$

D 11.

The differentiable function f on [–2, 2] is shown in the graph to the right. For how many values of x on [–2, 2] is the mean-value theorem satisfied?

A. 0 B. 1 C. 2
D. 3 E. 4



C 12.

(Calculator Active) How many values of c satisfies the Mean-Value Theorem for $f(x)=xe^{x^2}$ on [–1, 1]?

A. none B. 1 C. 2 D. 3 E. 4

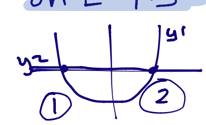
$f'(x)=\dfrac{f(1)-f(-1)}{1-(-1)}$

$(1)e^{x^2}+(x)(e^{x^2})(2x)=\dfrac{e-(-e)}{2}$

$e^{x^2}+2x^2e^{x^2}=\dfrac{\cancel{2}e}{\cancel{2}}$

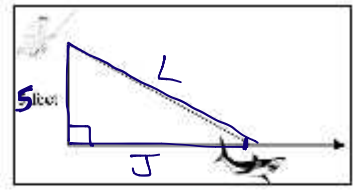
$\underbrace{e^{x^2}+2x^2e^{x^2}-e}_{y1}=\underbrace{0}_{y2}$

count x-intercepts on [-1,1]



E 13.

Jaws the shark is swimming horizontally 5 feet below the surface (see figure to the right) when he grabs onto some bait from a fisherman. If the length of the fishing line is 13 feet and Jaws swims at 20 feet per second, approximately how fast does the length of the fishing line increase at that moment?



A. $12\frac{\text{ft}}{\text{sec}}$ B. $7.7\frac{\text{ft}}{\text{sec}}$ C. $31.2\frac{\text{ft}}{\text{sec}}$ D. $37\frac{\text{ft}}{\text{sec}}$ E. $18.5\frac{\text{ft}}{\text{sec}}$

$\boxed{L=13 \quad J=12}$

$\frac{dJ}{dt}=20$ $\quad L^2=5^2+J^2$ "pos"

$\frac{dL}{dt}=?$

$\frac{d}{dt}: 2L\frac{dL}{dt}=2J\frac{dJ}{dt}$

$L\frac{dL}{dt}=J\frac{dJ}{dt}$

when $L=13$:

$13^2-5^2=J^2$

$J=\sqrt{144}$

$J=12$

when $L=13$: $(13)\frac{dL}{dt}=12(20)$

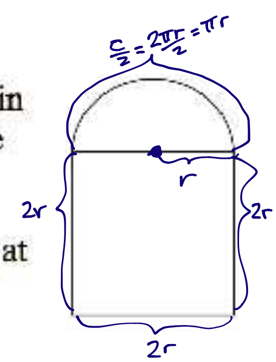
$\frac{dL}{dt}=\frac{240}{13}$ ft/sec

$\frac{dL}{dt}=18.46153846\ldots$ ft/sec

$\frac{dL}{dt}\approx 18.5$ ft/sec

C 14.

Jerry creates a computer graphic made up of a semicircle on top of a square as shown in the figure to the right. He drags the lower right corner of the square to make the entire graphic larger. When the radius of the circle is 2 inches, the area of the graphic is increasing at the rate of $10\frac{\text{inch}^2}{\text{sec}}$. How fast is the perimeter of the graphic increasing at the same time in $\frac{\text{inch}}{\text{sec}}$?



$\boxed{r=2}$ $\quad \frac{dA}{dt}=10, \quad \frac{dP}{dt}=?$

A. $\frac{15}{4}$ B. $\frac{15}{2}$ C. $\frac{5(6+\pi)}{8+\pi}$ D. $\frac{5(6+\pi)}{2(4+\pi)}$ E. 5

$A=(2r)^2+\frac{\pi}{2}r^2$

$A=4r^2+\frac{\pi}{2}r^2$

$\frac{d}{dt}: \frac{dA}{dt}=8r\frac{dr}{dt}+\pi r\frac{dr}{dt}$

when $r=2$: $10=8(2)\frac{dr}{dt}+\pi(2)\frac{dr}{dt}$

$10=\frac{dr}{dt}(16+2\pi)$

$\frac{dr}{dt}=\frac{10}{16+2\pi}$

$\boxed{\frac{dr}{dt}=\frac{5}{8+\pi}}$

$P=2r+2r+2r+2\pi r$

$P=6r+2\pi r$

$\frac{d}{dt}: \frac{dP}{dt}=6\frac{dr}{dt}+2\pi\frac{dr}{dt}$

$\frac{dP}{dt}=(6+2\pi)\frac{dr}{dt}$

when $r=2$: $\frac{dP}{dt}=(6+2\pi)\left(\frac{5}{8+\pi}\right)$

$=\frac{5(6+2\pi)}{8+\pi}$

E 15.

Suppose $f(x)$ is differentiable everywhere and $f(-2)=-5$ and $f'(x)\le 5$ for all values of x. Using the Mean-Value Theorem, what is the largest possible value of $f(8)$?

A) 35 E) 45 I) 55 O) 65 U) 75

$$f'(x)=\frac{f(8)-f(-2)}{8-(-2)}\le 5$$

$$\frac{f(8)-(-5)}{10}\le 5$$

$$f(8)+5\le 50$$

$$f(8)\le 45$$

P 16.

If $f'(x)=\underset{\text{neg}}{-}\underbrace{(\ln x-1)^2}_{\text{pos}}\underbrace{(\sin x-2)}_{\text{neg}}\boxed{(x-3)}$, $x>0$, $f(x)$ has which of the following relative extrema?

I. Relative minimum at $x=e$ II. Relative maximum at $x=2$ III. Relative minimum at $x=3$

M) I only N) II only P) III only R) I and III only S) none of these

$f'(x)=0$

$\ln x-1=0$
$\ln x=1$
$\boxed{x=e}$

$\sin x-2=0$
$\sin x=2$
No soln

$x-3=0$
$\boxed{x=3}$

x	1	e	2.9	3	4
f'	$-$		$-$		$+$

*the sign of f' is determined exclusively by the sign of $x-3$.

N 17.

If $f'(x)=3x^{5/3}-30x^{2/3}$, for what values of x is f concave up?

B) $(0,4)$ H) $(4,\infty)$ N) $(-\infty,0)\cup(4,\infty)$ T) $(-\infty,0)\cup(2,\infty)$ V) $(0,2)$

$$f'(x)=3x^{5/3}-30x^{2/3}$$

$$f''(x)=5x^{2/3}-20x^{-1/3}$$

$$=5x^{-1/3}[x-4]$$

$$f''(x)=\frac{5(x-4)}{\sqrt[3]{x}}$$

$f''(x)=0$
$x-4=0$
$\boxed{x=4}$

$f''=\text{DNE}$
$\sqrt[3]{x}=0$
$\boxed{x=0}$

x	-1	0	1	4	5
f''	$+$		$-$		$+$

18.

. Suppose f is a function defined on $[-8, 8]$ given by $f(x) = kx^{1/3} - x^{2/3}$, where k is a positive constant less than 2.

$D_f : \mathbb{R}$

$f(x) = kx^{1/3} - x^{2/3}$ $\quad 0 < k < 2$

a. Show that $f'(x) = \dfrac{k - 2x^{1/3}}{3x^{2/3}}$

$f(x) = kx^{1/3} - x^{2/3}$

$f'(x) = \frac{k}{3}x^{-2/3} - \frac{2}{3}x^{-1/3}$ ✓1

$= \frac{1}{3}x^{-2/3}\left[k - 2x^{1/3}\right]$

$f'(x) = \dfrac{k - 2x^{1/3}}{3x^{2/3}}$ ✓2

b. For what value of x, written as a function of k, is $f(x)$ increasing? Justify your answer.

$f'(x) = \dfrac{k - 2x^{1/3}}{3x^{2/3}} = \dfrac{k - 2\sqrt[3]{x}}{3\sqrt[3]{x^2}}$

$f'(x) = 0$

$k - 2x^{1/3} = 0$

$k = 2x^{1/3}$

$x^{1/3} = \frac{k}{2}$

$x = \left(\frac{k}{2}\right)^3$

$\boxed{x = \frac{k^3}{8}}$ ✓3

where

$0 < \frac{k^3}{8} < 1$

since $0 < k < 2$

$f' = \text{DNE}$

$3x^{2/3} = 0$

$x^{2/3} = 0$

$\boxed{x = 0}$ ✓4

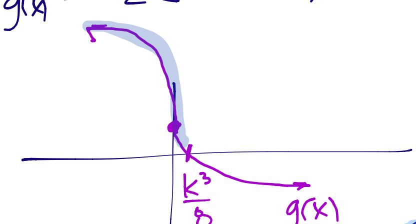
x	-1	0	m	$\frac{k^3}{8}$	1	ω
f'	$+$		$+$		$-$	

* see graph below

Since $3\sqrt[3]{x^2} > 0 \ \forall x \neq 0$, the sign of f' is determined exclusively by the numerator,

$g(x) = k - 2\sqrt[3]{x}$.

$g(x) = -2\sqrt[3]{x} + k$, $0 < k < 2$



$\boxed{\text{So, } f \text{ is increasing for } x < \frac{k^3}{8}}$ ✓5

c. Find the range of *f* on [–8, 8]. Show the analysis that leads to your answer.

$f(x) = kx^{1/3} - x^{2/3}$

$f(x) = k\sqrt[3]{x} - (\sqrt[3]{x})^2$

$f(-8) = -2k - 4$ ← MIN

$f(8) = 2k - 4$

$f(0) = 0$

$f\left(\frac{k^3}{8}\right) = k\left(\frac{k}{2}\right) - \left(\frac{k}{2}\right)^2$

$= \frac{k^2}{2} - \frac{k^2}{4}$

$= \frac{k^2}{4}$ ← MAX

$\boxed{R_f : \left[-2k-4, \frac{k^2}{4}\right]}$ √6

d. At $x = 1$, find the values of k for which $f(x)$ is concave up.

$f'(x) = \frac{k}{3}x^{-2/3} - \frac{2}{3}x^{-1/3}$

$f''(x) = -\frac{2k}{9}x^{-5/3} + \frac{2}{9}x^{-4/3}$

$f''(1) = -\frac{2}{9}k + \frac{2}{9} = 0$

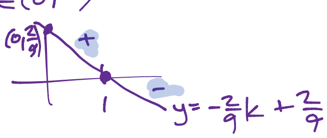
$\frac{2}{9} = \frac{2}{9}k$

$k = 1$

k	$\frac{9}{10}$	$\frac{9}{8}$
f''	+	—

, $k \in (0, 2)$

or



So, f is concave up for $\boxed{0 < k < 1}$

√7