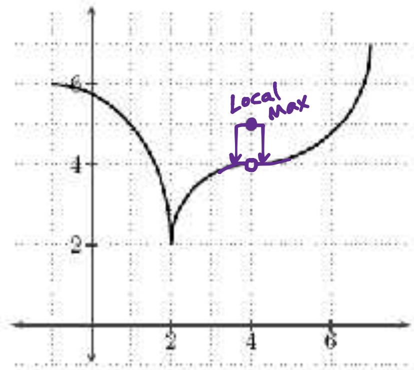
Name KEY Date Reindeer's favorite game

TEST: 3.1-3.6, NO CALCULATOR

Part I: Multiple Choice: Put the letter in the letter place.



D 1. The graph of a function $f(x)$ is shown above. Which of the following properties does f NOT have?

(A) $\lim_{x\to 4} f(x) = 4$ (B) $f'(x) < 0$ on $(-1, 2)$ dec (C) $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$

(D) differentiable at $x = 2$ No, cusp (E) local maximum at $x = 4$

A 2. Let f be the function given by $f(x) = x^3 - 6x^2$. The graph of f is concave up when

(A) $x > 2$ (B) $x < 2$ (C) $0 < x < 4$ (D) $x < 0$ or $x > 4$ only (E) $x > 6$ only

$f'' > 0$, $f' = 3x^2 - 12x$

$f'' = 6x - 12 = 0$

$\boxed{x = 2}$

x	1	2	3
f''	$-$		$+$

A 3. The value of k for which $f(x) = x + \frac{k}{x}$ has a relative minimum at $x = 3$ is

(A) 9 (B) 6 (C) -3 (D) -6 (E) -9

$f'(x) = 1 - \frac{k}{x^2}$

$f'(3) = 1 - \frac{k}{9} = 0$

$\frac{k}{9} = 1$

$\boxed{k = 9}$

so, $f' = 1 - \frac{9}{x^2}$

check: $x = 3$

x	2	4
f'	$-$	$+$

B 4. If $y+8=2x$, what is the minimum value of the product xy?

(A) -16 (B) -8 (C) -4 (D) 0 (E) 2

$P = xy$, $y = 2x-8$

$P = x(2x-8)$

$P = 2x^2 - 8x$

$P' = 4x - 8 = 0$

$x = 2$

x	1	2	3
P'	$-$		$+$

Since $P' < 0\ \forall x < 2$ and $P' > 0\ \forall x > 2$, P is minimized when $x = 2$.

So min product is $P(2) = 2(2(2)-8)$

$P(2) = 2(4-8)$

$P(2) = 2(-4)$

$\boxed{P(2) = -8}$

C 5. For $x > 0$, f is a function such that $f'(x) = \dfrac{\ln x}{x}$ and $f''(x) = \dfrac{1-\ln x}{x^2}$. Which of the following is true?

(A) f is ~~decreasing for~~ $x > 1$, and the graph of f is concave down for $x > e$

(B) f is ~~decreasing for~~ $x > 1$, and the graph of f is concave up for $x > e$

(C) f is increasing for $x > 1$, and the graph of f is concave down for $x > e$

(D) f is increasing for $x > 1$, and the graph of f is ~~concave up for~~ $x > e$

(E) f is decreasing for $0 < x < 1$, and the graph of f is concave down for $0 < x < e^{3/2}$

$f' = 0$

$\ln x = 0$

$\boxed{x = 1}$

x	$1/2$	1	2
f'	$-$		$+$

$f'' = 0$

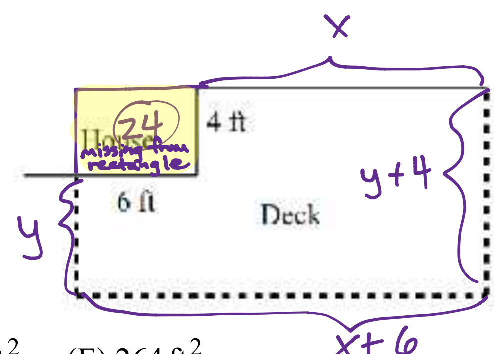
$1 - \ln x = 0$

$\ln x = 1$

$\boxed{x = e}$

x	1	e	e^2
f''	$+$		$-$

B 6. I have volunteered this class to build a deck at my house (as shown in the figure at the right). My wife has 40 linear feet of decorative railing that is to be used around the deck (shown by the dashed lines). With this in mind, what is the maximum deck area that you excellent people can achieve to accommodate my wife's wishes? (Assume you all are using all the decorative railing, neither wasting any nor buying any more.)



(A) 176 ft^2 (B) 218 ft^2 (C) 240.5 ft^2 (D) 242 ft^2 (E) 264 ft^2

Constraint (Dotted)

$40 = y + x + 6 + y + 4$

$40 = 2y + x + 10$

$30 = x + 2y$

$\boxed{x = 30 - 2y}$

$A = \underbrace{(x+6)(y+4)}_{\text{total area}} - \underbrace{24}_{\text{yellow Notch}}$

$A = (30 - 2y + 6)(y+4) - 24$

$A = (36 - 2y)(y+4) - 24$

$A' = (-2)(y+4) + (36-2y)(1)$

$A' = -2[y + 4 - 18 + y]$

$A' = -2[2y - 14]$

$A' = -4[y - 7] = 0$

$\boxed{y = 7}$

A' 28

7

$A' > 0, \forall x < 7$

$A' < 0, \forall x > 7$, so $y = 7$ maximizes A.

So,

$A(7) = (36 - 2(7))(7+4) - 24$

$= (36-14)(11) - 24$

$= (22)(11) - 24$

$= 22(10+1) - 24$

$= 220 + 22 - 24$

$= 220 - 2$

$= \boxed{218\ ft^2}$

* Note: There are many, many, many ways to set up and work this problem.

E 7. What are the coordinates of the inflection point on the graph of $y=(x+1)\arctan x$?

(A) $(-1,0)$ (B) $(0,0)$ (C) $(0,1)$ (D) $\left(1,\frac{\pi}{4}\right)$ (E) $\left(1,\frac{\pi}{2}\right)$

$y' = (1)\arctan x + (x+1)\left(\frac{1}{1+x^2}\right)$

$y' = \arctan x + \frac{1+x}{1+x^2}$

$y'' = \frac{1}{1+x^2} + \frac{(1+x^2)(1)-(1+x)(2x)}{(1+x^2)^2}$

$y'' = \frac{1+x^2}{(1+x^2)^2} + \frac{1+x^2-2x-2x^2}{(1+x^2)^2}$

$y'' = \frac{1+\cancel{x^2}+1+\cancel{x^2}-2x-\cancel{2x^2}}{(1+x^2)^2}$

$y'' = \frac{2-2x}{(1+x^2)^2} = 0$

when $2-2x=0$, $\boxed{x=1}$

x	0	1	2
y''	+		−

So inflection pt at $(1, f(1)) =$ $(1, 2\arctan 1)$ $(1, 2\cdot\frac{\pi}{4})$ $\boxed{(1,\frac{\pi}{2})}$

D 8. Let f be a function with a second derivative given by $f''(x)=x^2(x-3)(x-6)$. What are the x-coordinates of the points of inflection of the graph of f?

(A) 0 only (B) 3 only (C) 0 and 6 only (D) 3 and 6 only (E) 0, 3, and 6 only

p.i.v.s: $x=0,3,6$

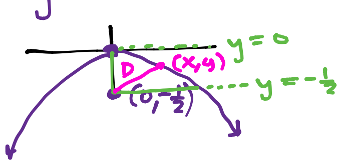
sign change in f'' only at $x=3$, $x=6$

B 9. The point on the curve $x^2+2y=0$ that is nearest the point $\left(0,-\frac{1}{2}\right)$ occurs where y is

(A) $\frac{1}{2}$ (B) 0 (C) $-\frac{1}{2}$ (D) -1 (E) $-\frac{1}{4}$

$2y = -x^2$

$y = -\frac{1}{2}x^2$



$D=\sqrt{(x-0)^2+(y+\frac{1}{2})^2}$

$D=\sqrt{x^2+(-\frac{1}{2}x^2+\frac{1}{2})^2}$

Let $R = x^2+(\frac{1}{2}-\frac{1}{2}x^2)^2$

$R' = 2x + 2(\frac{1}{2}-\frac{1}{2}x^2)^1(-x)=0$

$2x-2x(\frac{1}{2}-\frac{1}{2}x^2)=0$

$2x-x+x^3=0$

$x+x^3=0$

$x(1+x^2)=0$

$\boxed{x=0}$ $1+x^2=0$ No Solution

$R''=1+3x^2>0\ \forall x$ (R is cc up $\forall x$)

so $x=0$ minimizes D.

when $x=0$,

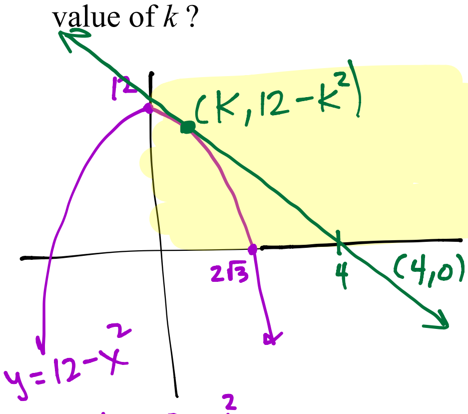
$y=-\frac{1}{2}(0^2)$

$\boxed{y=0}$

Part II: Free Response

10. (1990 BC3) Let $f(x) = 12 - x^2$ for $x \geq 0$ and $f(x) \geq 0$. → Quadrant I

(a) The line tangent to the graph of f at the point $(k, f(k))$ intersects the x-axis at $x = 4$. What is the value of k?



x-ints: $12 - x^2 = 0$

$x^2 = 12$

$x = \pm\sqrt{12}$

$x = \pm 2\sqrt{3}$

$x = \sqrt{12}$ or $2\sqrt{3}$

m_1: ALGEBRA

$m = \dfrac{12-k^2-0}{k-4} = \dfrac{\Delta y}{\Delta x}$

$\boxed{m = \dfrac{12-k^2}{k-4}}$ ✓1

m_2: CALCULUS

$y' = -2x$ ✓2 & ✓3

$y'(k) = \boxed{-2k = m}$

So $\boxed{\dfrac{12-k^2}{k-4} = -2k}$

$12 - k^2 = -2k(k-4)$

$12 - k^2 = -2k^2 + 8k$

$k^2 - 8k + 12 = 0$

$(k-6)(k-2) = 0$

So $k = 6$ or $k = 2$

$k \neq 6$ since $x = 6$ is not a common pt on the parabola & tangent line,

So $\boxed{k = 2}$ ✓4

or $y = -2x$

$y(k) = -2k = m$

pt: $(4,0)$

eq: $y = 0 - 2k(x-4)$

But $f(k) = y(k)$, so

$12 - k^2 = -2k(x-4)$

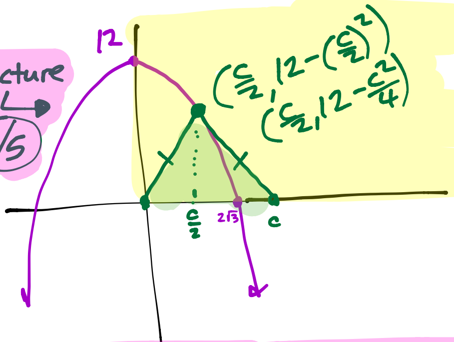
$k^2 - 8k + 12 = 0$

$(k-6)(k-2) = 0$

$\cancel{k=6}$ $\boxed{k=2}$

(b) An isosceles triangle whose base is the interval from $(0,0)$ to $(c,0)$ (but not necessarily contained within the x-intercept of the graph of f) has its vertex on the graph of f. For what value of c does the triangle have its maximum area? Justify your answer.

Picture → ✓5



width of isosceles from $x=0$ to $x=c$ is $c - 0 = c$, so the height of the triangle occurs at $x = \frac{c}{2}$

Max Area:

$A = \frac{1}{2}bh$ ✓6

$A = \left(\frac{1}{2}\right)(c)\left(12 - \frac{c^2}{4}\right)$

$A = 6c - \frac{1}{8}c^3$, $c \in (0, 4\sqrt{3})$ ← twice the x-int of $y = 12 - x^2$

$A' = 6 - \frac{3}{8}c^2 = 0$ ✓7

$\frac{3}{8}c^2 = 6$

$c^2 = (6)\left(\frac{8}{3}\right)$

$c^2 = 16$

$\cancel{c = -4}$ not in QI, $\boxed{c = 4}$ ✓8

Justification ✓9

$c = 4$ maximizes A since

(I) $A' > 0 \ \forall x \in (0,4)$ & *Lower bound needed

$A' < 0 \ \forall x \in (4, 4\sqrt{3})$ *upper bound may be left off: $x > 4$

or (II) $A'' = -\frac{3}{4}c < 0 \ \forall x \in (0, 4\sqrt{3})$

or (III) $A(0) = 0$

$A(4) = 16$ ← MAX

$A(4\sqrt{3}) = 0$

1990 BC3
Solution

(a) $f(x) = 12 - x^2$; $f'(x) = -2x$

slope of tangent line at

$(k, f(k)) = -2k$

line through $(4,0)$ & $(k, f(k))$ has slope

$\dfrac{f(k) - 0}{k-4} = \dfrac{12-k^2}{k-4}$

so $-2k = \dfrac{12-k^2}{k-4}$; $k^2 - 8k + 12 = 0$

$k = 2$ or $k = 6$ but $f(6) = -24$

so 6 is not in the domain.

$k = 2$

(b) $A = \dfrac{1}{2}c \cdot f\left(\dfrac{c}{2}\right) = \dfrac{1}{2}c\left(12 - \dfrac{c^2}{4}\right)$

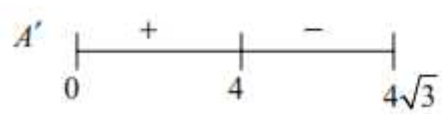
$= 6c - \dfrac{c^3}{8}$ on $\left[0, 4\sqrt{3}\right]$

$\dfrac{dA}{dc} = 6 - \dfrac{3c^2}{8}$; $6 - \dfrac{3c^2}{8} = 0$ when c = 4.

Candidate test

c	A
0	0
4	16 ← Max
$4\sqrt{3}$	0

First derivative



second derivative

$\left.\dfrac{d^2A}{dc^2}\right|_{c=4} = -3 < 0$ so $c = 4$ gives a relative max.

$c = 4$ is the only critical value in the domain interval, therefore maximum