Name________________________________ Date______________________ Period_____________

Calculus Test: 2.1 to 3.1. No Calculator

MULITPLE CHOICE: Show all work on attached paper. Put the CAPITAL letter in the blank.

__B__ 1. If $f(3)=2$, $g(3)=-\frac{3}{2}$, $f'(3)=-2$, $g'(3)=5$, and $h(x)=\left[f(x)+2g(x)\right]^3$, find $h'(3)$.

(A) -24 (B) 24 (C) 1 (D) -1 (E) 42

$h'(x)=3[f+2g]^2(f'+2g')$

$h'(3)=3[2+2(-\frac{3}{2})]^2(-2+2(5))$

$=3(2-3)^2(-2+10)$

$=3(-1)^2(8)$

$=24$

__A__ 2. If $f(x)=\sqrt{\tan\left(2x-\frac{3\pi}{4}\right)}$, find $\lim\limits_{x\to\pi/2}\dfrac{f(x)-f(\pi/2)}{x-\pi/2}$ $= f'(\frac{\pi}{2})$ (alternate form)

(A) 2 (B) -2 (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$ (E) 4

$f'(x)=\frac{1}{2}\left(\tan(2x-\frac{3\pi}{4})\right)^{-1/2}\cdot\sec^2(2x-\frac{3\pi}{4})\cdot(2)$

$f'(x)=\dfrac{(\sec(2x-\frac{3\pi}{4}))^2}{\sqrt{\tan(2x-\frac{3\pi}{4})}}$

$f'(\frac{\pi}{2})=\dfrac{(\sec(\pi-\frac{3\pi}{4}))^2}{\sqrt{\tan(\pi-\frac{3\pi}{4})}}$

$f'(\frac{\pi}{2})=\dfrac{(\sec\frac{\pi}{4})^2}{\sqrt{\tan\frac{\pi}{4}}}$

$f'(\frac{\pi}{2})=\dfrac{(\sqrt{2})^2}{\sqrt{1}}$

$f'(\frac{\pi}{2})=2$

__A__ 3. If $x^2+y^2=k$ where k is a non-zero constant, in which quadrants is $\dfrac{d^3y}{dx^3}<0$?

(A) I and III only (B) I and II only (C) III and IV only (D) II and IV only (E) all quadrants

$\frac{d}{dx}$: $2x+2yy'=0$

$y'=-\frac{x}{y}$

$\frac{d^2}{dx^2}$: $y''=\dfrac{y(-1)-(-x)y'}{y^2}$

$y''=\dfrac{\left(-y+x(-\frac{x}{y})\right)}{y^2}\left(\frac{y}{y}\right)$

$y''=\dfrac{-y^2-x^2}{y^3}$

$\frac{d^3}{dx^3}$: $y'''=\dfrac{(y^3)(-2yy'-2x)-(-y^2-x^2)(3y^2y')}{y^6}$

$y'''=\dfrac{(y^3)(-2y(-\frac{x}{y})-2x)+(y^2+x^2)(3y^2(-\frac{x}{y}))}{y^6}$

$y'''=\dfrac{(y^3)(2x-2x)+(y^2+x^2)(-3xy)}{y^6}$

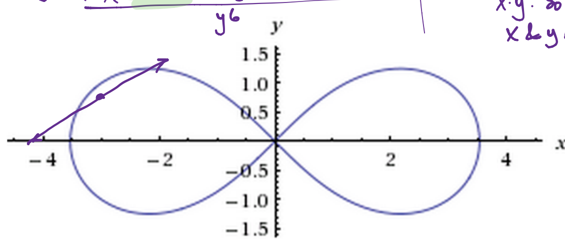
$y'''=\dfrac{d^3y}{dx^3}=\dfrac{-3xy(x^2+y^2)}{y^6}$

* since $\dfrac{-3(x^2+y^2)}{y^6}<0\ \forall x, \forall y\neq 0$

the sign of y''' is determined by $x\cdot y$. So, for $y'''<0$, $xy>0$ and x & y must be the same sign.

$x>0, y>0 \to$ QI

$x<0, y<0 \to$ QIII



__D__ 4. The figure above shows the graph of of $2\left(x^2+y^2\right)^2=25\left(x^2-y^2\right)$. Find the y-intercept of the tangent line to the above graph at $(-3,1)$.

(A) $\left(0,\frac{14}{13}\right)$ (B) $\left(0,\frac{5}{2}\right)$ (C) $(0,10)$ (D) $\left(0,\frac{40}{13}\right)$ (E) $(0,3)$

$\frac{d}{dx}$: $4(x^2+y^2)\cdot(2x+2yy')=25(2x-2yy')$

At $(-3,1)$: $4(10)(-6+2y')=25(-6-2y')$

$-240+80y'=-150-50y'$

$130y'=90$

$y'=\frac{90}{130}$

$y'=\frac{9}{13}=m$

eq: $y=1+\frac{9}{13}(x+3)$

y-int Let $x=0$: $y=1+\frac{9}{13}(3)$

$y=\frac{13}{13}+\frac{27}{13}$

$y=\frac{40}{13}$

E 5. If $f(x)=(\sin x)^{\ln x}$, then $f'(x)=$

(A) $\dfrac{\ln(\sin x)\cdot(\sin x)^{\ln x}}{x}$ (B) $\dfrac{\ln(\sin x)}{x}+\ln x(\cot x)$ (C) $(\ln x)(\sin x)^{\ln x-1}$

(D) $\dfrac{(\sin x)^{\ln x}}{x}$ (E) $\left(\dfrac{\ln(\sin x)}{x}+\ln x(\cot x)\right)(\sin x)^{\ln x}$

Log Diff $(\text{var})^{\text{var}}$
Let $y=f(x)$

$\ln y=\ln x\cdot\ln(\sin x)$

$\frac{d}{dx}:\ \frac{1}{y}\cdot y'=\left(\frac{1}{x}\right)\ln(\sin x)+(\ln x)\left(\frac{1}{\sin x}\right)(\cos x)$

$y'=\left[\frac{\ln(\sin x)}{x}+(\ln x)(\cot x)\right]\cdot(\sin x)^{\ln x}$

B * At any point of tangency, a function & its tangent line share a y-value & a slope value!

6. The line $y_1=16x+16$ is tangent to the graph of $y_2=x^3+4x$ at

I. $x=2$
II. $x=-2$
III. $x=-4$

$y_1'=16$ $y_2'=3x^2+4$

$3x^2+4=16$
$3x^2=12$
$x^2=4$
$x=\pm 2$

* check y-values
$y_1(2)=32+16=48$
$y_2(2)=8+8=16$
$48\neq 16$

$y_1(-2)=-32+16=-16$
$y_2(-2)=-8-8=-16$
$-16=-16$ ✓
tangent at $x=-2$.

(A) I only (B) II only (C) II and III only (D) I and III only (E) I, II, and III

C 7. If $f(x)=3\cos(x)+e^{\pi-x}$, $f(\pi)=-2$, and $f(g(x))=x=g(f(x))$, then what is the value of $g'(-2)$?

(A) $-3\sin(-2)-e^{\pi 2}$ (B) 1 (C) -1 (D) $\dfrac{1}{-3\sin(-2)-e^{\pi 2}}$ (E) $-\dfrac{1}{2}$

$g: (-2,\pi)$
$f: (\pi,-2)$
so $g'(-2)=\dfrac{1}{f'(\pi)}$,

$f'(x)=-3\sin x-e^{\pi-x}$
$f'(\pi)=-3\sin\pi-e^{\pi-\pi}$
$=0-e^0$
$f'(\pi)=-1$
so $\frac{1}{f'(\pi)}=-1$ too!

A 8. If $f(x)=\ln\sqrt[5]{|\cos x|}$, find $f'(x)$.

(A) $-\dfrac{1}{5}\tan x$ (B) $\dfrac{1}{5}|\tan x|$ (C) $-\dfrac{1}{5}\cot x$ (D) $\dfrac{1}{(\cos x)^{1/5}}$ (E) $\dfrac{-\sin x}{(\cos x)^{1/5}}$

"Simplify early & often, especially when you have logs." (And we got 'em here!!)

$f(x)=\frac{1}{5}\ln|\cos x|$
$f'(x)=\frac{1}{5}\left(\frac{1}{\cos x}\right)\cdot(-\sin x)$
$f'(x)=-\frac{1}{5}\tan x$

D 9. Let $h(x)=e^{f(3x)}$. If $f(3)=-2$ and $h'(1)=e^2$, find $f'(3)$.

(A) e^4 (B) $3e^2$ (C) e^2 (D) $\frac{e^4}{3}$ (E) $\frac{e^2}{3}$

$h'(x) = e^{f(3x)} \cdot f'(3x) \cdot 3$

$h'(1) = e^{f(3)} \cdot f'(3) \cdot 3$

so $e^2 = e^{-2} \cdot f'(3) \cdot 3$

$f'(3) = \frac{e^2}{3e^{-2}}$

$f'(3) = \frac{e^4}{3}$

E 10. If $f(x)=2^x-\ln 2\cdot \log_2 x+e^{2\ln x}$, what is the slope of the tangent line to $f(x)$ at $x=1$?

$e^{\ln x^2} = x^2$

(A) $\ln(4)$ (B) $\ln\left(\frac{4}{e}\right)$ (C) $-\ln(4e)$ (D) $-\ln(4)$ (E) $\ln(4e)$

$f(x) = 2^x - \ln 2 \cdot \log_2 x + x^2$

$f'(x) = 2^x \cdot \ln 2 - \cancel{\ln 2}\left(\frac{1}{x \cdot \cancel{\ln 2}}\right) + 2x$

$f'(1) = 2 \cdot \ln 2 - 1 + 2$

$f'(1) = \ln 2^2 + 1$, $1 = \ln e$

$f'(1) = \ln 4 + \ln e$

$f'(1) = \ln(4e)$

B 11. The graph of $g(x)=\frac{e-\ln 2x}{x}$ has a horizontal tangent line at what x-value?

$g'(x)=0$

(A) $\frac{1}{2}e^{-e-1}$ (B) $\frac{1}{2}e^{e+1}$ (C) e^{e+1} (D) e^{-e-1} (E) $\frac{1}{2}e^{e-1}$

$g'(x) = \frac{(x)\left(-\frac{2}{2x}\right) - (e - \ln 2x)(1)}{x^2}$

$g'(x) = \frac{-1 - e + \ln 2x}{x^2} = 0$

when $-1 - e + \ln 2x = 0$

$\ln 2x = e + 1$

$e^{\ln 2x} = e^{e+1}$

$2x = e^{e+1}$

$x = \frac{1}{2}e^{e+1}$

A 12. The graph of the equation $x^2+4x=6+3y+3y^{-1}$ passes through many points, including the following 6: $(-6,1)$, $(2,1)$, $(0,-1)$, $(-2,-3)$, $\left(-2,-\frac{1}{3}\right)$, and $(-4,-1)$. These 6 points are either points of horizontal tangent lines (H), vertical tangent lines (V), or neither. How many of each type of tangent lines does this graph have at these points?

V V V H H V

(A) 2H, 4V (B) 4H, 2V (C) 3H, 2V (D) 2H, 2V (E) 2H, 0V

$\frac{d}{dx}: 2x + 4 = 0 + 3y' - 3y^{-2} \cdot y'$

$2x + 4 = y'(3 - 3y^{-2})$

$y' = \frac{2x+4}{3 - \frac{3}{y^2}}$

2H, 4V

Horz Tangents

$y' = 0$

$2x + 4 = 0$

$\boxed{x = -2}$

Vert Tangents

$y' = \frac{\neq 0}{0}$

$3 - \frac{3}{y^2} = 0$

$3 = \frac{3}{y^2}$

$y^2 = 1$

$\boxed{y = \pm 1}$

B 13. A baby unicorn is moving along a horizontal line and has velocity $v(t)=\ln\left(t-t^2\right)$ for all values $0<t<1$. For what value(s) of t is the speed of the cute, baby unicorn decreasing?

Need $v(t)$ & $v'(t)$ to be opposite signs.

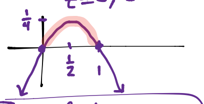
(A) $0<t<1$ (B) $0<t<\frac{1}{2}$ (C) $\frac{1}{2}<t<1$ (D) $\frac{1}{4}<t<\frac{3}{4}$ (E) no such values

Domain of $v(t)$: $(0,1)$

$t-t^2>0$

$t(1-t)>0$

$t=0, t=1$



$\boxed{\text{Since } \ln t<0 \ \forall t\in(0,1) \quad v(t)<0 \ \forall t \in (0,1)}$

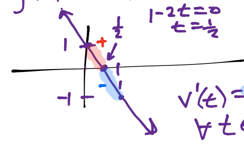
$v'(t)=a(t)=\frac{1}{t-t^2}(1-2t)$

$v'(t)=\frac{1-2t}{t-t^2}$ since $t-t^2>0$ for all $t\in(0,1)$, the sign of $v'(t)$ is determined by $1-2t$.

$y=1-2t$ for $t\in(0,1)$

$1-2t=0$

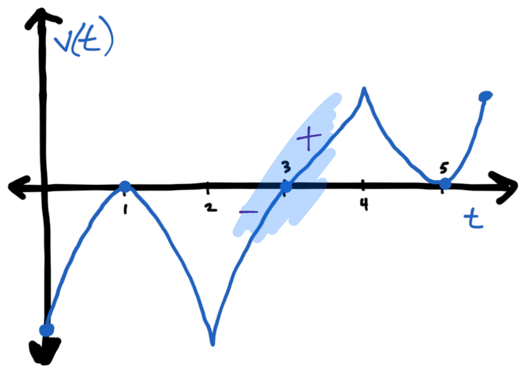
$t=\frac{1}{2}$



$v'(t)=a(t)>0$

$\forall t\in\left(0,\frac{1}{2}\right)$

$\therefore$ since $v(t)<0$ for $t\in\left(0,\frac{1}{2}\right)$ and $v'(t)=a(t)>0$ for $t\in\left(0,\frac{1}{2}\right)$, the baby unicorn's speed i DECREASING for all $t\in\left(0,\frac{1}{2}\right)$.



E 14. A big nerd is walking along down a straight road towards his compass with a velocity function $v(t)$ as shown in the figure above. For what values of t does the nerd change direction?

Need $v(t)$ to change signs.

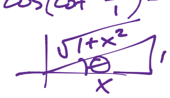
(A) 1, 2, 4, and 5 only (B) 1 and 5 only (C) 2 and 4 only (D) 1, 2, and 5 only (E) 3 only

since $v(t)$ changes from negative to positive at $t=3$, the big nerd changes directions at $t=3$.

E 15. If $f(x)=\cos\left(\cot^{-1}x\right)$, find $f'(x)$.

(A) $\frac{-1}{\sqrt{1+x^2}}$ (B) $\frac{1}{\sqrt{1+x^2}}$ (C) $\frac{1}{\sqrt{\left(1-x^2\right)^3}}$ (D) $\frac{1}{\sqrt{1-x^2}}$ (E) $\frac{1}{\sqrt{\left(1+x^2\right)^3}}$

$\cos\left(\cot^{-1}\frac{x}{1}\right)=\frac{x}{\sqrt{1+x^2}}=x(1+x^2)^{-1/2}$



$f'(x)=(1)(1+x^2)^{-1/2}+(x)\left(-\frac{1}{2}\right)(1+x^2)^{-3/2}(2x)$

$f'(x)=(1+x^2)^{-3/2}\left[(1+x^2)-x^2\right]$

$f'(x)=\frac{1}{\sqrt{(1+x^2)^3}}$

D 16. Find the equation of the normal line to $g(x)=\arctan(\ln x)$ at $x=e$.

(A) $y=-2e(x-e)$ (B) $y=\frac{\pi}{4}-2(x-e)$ (C) $y=-2(x-e)$ (D) $y=\frac{\pi}{4}-2e(x-e)$ (E) $y=\frac{\pi}{2}-2e(x-e)$

$g'(x)=\frac{1}{1+(\ln x)^2}\cdot\left(\frac{1}{x}\right)$

$g'(e)=\frac{1}{1+1}\left(\frac{1}{e}\right)$

$g'(e)=\frac{1}{2e}=m_{tangent}$

$\boxed{m_{normal}=-2e}$

$g(e)=\arctan(\ln e)$

$=\arctan(1)$

$g(e)=\frac{\pi}{4}$

$\boxed{pt: \left(e,\frac{\pi}{4}\right)}$

eq: $y=\frac{\pi}{4}-2e(x-e)$