KEY

Name________________________________ Date__________ Favorite Cold Yam Temperature_______

AP Calculus TEST: 2.1-2.7, NO CALCULATOR

Part I: Multiple Choice—Put the correct CAPITAL letter in the space to the left of each question. SHOW ALL WORK/INDICATE ALL METHODS on a SEPARATE SHEET OF PAPER. Number your scratch work.

__A__ 1. $\lim_{h\to 0} \dfrac{\dfrac{8}{\sqrt[3]{(\cancel{6+h})+2}} - \dfrac{8}{\sqrt[3]{6+2}}}{h} = f'(6)$, for $f(x) = \dfrac{8}{\sqrt[3]{x+2}}$,

$f(x) = 8(x+2)^{-1/3}$

$f'(x) = -\frac{8}{3}(x+2)^{-4/3} = -\dfrac{8}{3(\sqrt[3]{x+2})^4}$

$f'(6) = \dfrac{-8}{3(\sqrt[3]{8})^4} = \dfrac{-8}{3\cdot 16} = \dfrac{-1}{6}$

(A) $-\frac{1}{6}$ (B) $\frac{1}{6}$ (C) $-\frac{1}{2}$ (D) $\frac{1}{2}$ (E) DNE

__A__ 2. If $f(x) = \dfrac{x^2+c^2}{x^2-c^2}$ where c is a constant, then $f'(x) =$

$f' = \dfrac{(x^2-c^2)(2x)-(x^2+c^2)(2x)}{(x^2-c^2)^2}$

$f' = \dfrac{2x[\cancel{x^2}-c^2-\cancel{x^2}-c^2]}{(x^2-c^2)^2}$

$f' = \dfrac{2x(-2c^2)}{(x^2-c^2)^2} = \dfrac{-4c^2x}{(x^2-c^2)^2}$

(A) $\dfrac{-4c^2x}{(x^2-c^2)^2}$ (B) $\dfrac{-2c^2x}{(x^2-c^2)^2}$ (C) $\dfrac{4c^2x}{(x^2-c^2)}$ (D) $\dfrac{2c^2x}{(x^2-c^2)^2}$ (E) 1

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	-2	$\frac{5}{2}$	13
2	-3	5	5	$-\frac{1}{2}$

continuous, too! continuous

The table above gives the values for differentiable functions $f(x)$ and $g(x)$ & their derivatives at selected values. **Use the table to answer questions 3, 4, and 5.**

$h'(x) = 3(f(2x))^2\cdot f'(2x)\cdot 2$, $h'(1) = 6(f(2))^2\cdot f'(2)$

__D__ 3. If $h(x) = [f(2x)]^3$, use the table to find $h'(1)$. $= 6(-3)^2\cdot 5 = 6\cdot 9\cdot 5 = 30\cdot 9 = 270$

(A) 54 (B) -90 (C) 135 (D) 270 (E) $\frac{10}{3}$

__B__ 4. If $K(x) = \sqrt[3]{f(x)+2g(x)}$, find $K'(1)$. $= \dfrac{f'(1)+2g'(1)}{3(\sqrt[3]{f(1)+2g(1)})^2} = \dfrac{-2+2(13)}{3(\sqrt[3]{3+2(\frac{5}{2})})^2} = \dfrac{24}{3\cdot 4} = 2$

(A) -2 (B) 2 (C) $-\frac{3}{2}$ (D) $\frac{3}{2}$ (E) 0

$K'(x) = \frac{1}{3}(f(x)+2g(x))^{-2/3}\cdot(f'(x)+2g'(x))$

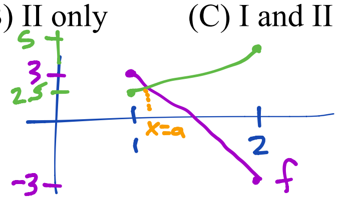
__E__ 5. For $1 \le x \le 2$, which of the following must be true?

✓ I. $g(c) = \frac{7}{2}$ for some $c \in (1,2)$ IVT $g(1) = \frac{5}{2} < 3 < 5 = g(2)$, so

✓ II. $g'(r) = 0$ for some $r \in (1,2)$ IVT g' is continuous $g'(2) = -\frac{1}{2} < 0 < 13 = g'(1)$, so yes, by IVT

✓ III. $f(a) = g(a)$ for some $a \in (1,2)$ IVT

(A) I only (B) II only (C) I and II only (D) II and III only (E) I, II, and III



$y = \frac{3}{f(x)}$, $y = \frac{3}{\cot x}$, $y = 3\tan x$

$\frac{dy}{dx} = 3\sec^2 x$

B 6. If $f(x) = \cot x$ and $\frac{3}{y} = f(x)$, find $\left.\frac{dy}{dx}\right|_{x=\frac{11\pi}{6}}$

$\left.\frac{dy}{dx}\right|_{x=\frac{11\pi}{6}} = 3\left[\sec\frac{11\pi}{6}\right]^2 = 3\left[\frac{2}{\sqrt{3}}\right]^2 = 3\left(\frac{4}{3}\right) = 4$

(A) 0 (B) 4 (C) 12 (D) $3\sqrt{3}$ (E) $-\sqrt{3}$

C 7. Let $f(x) = 2\sin x\cos x$ for $0 \le x < \pi$. Find all the values for which $f'(x) = 1$.

I. $x = \frac{\pi}{6}$

II. $x = \frac{5\pi}{6}$

III. $x = \frac{\pi}{3}$

$f(x) = \sin 2x$

$f'(x) = 2\cos 2x = 1$

$\cos 2x = \frac{1}{2}$

$2x = \frac{\pi}{3}$, $x = \frac{\pi}{6}$

$2x = \frac{5\pi}{3}$, $x = \frac{5\pi}{6}$

(A) I only (B) II only (C) I, II only (D) I, III only (E) II, and III only

C 8. If $h(x) = x^2 g'(x)$, where $g(x) = \frac{x + \sec x}{x}$, then what is the slope of the secant line on the graph of $h(x)$ for $x \in [\pi, 2\pi]$?

$g'(x) = \frac{(x)(1+\sec x\tan x) - (x+\sec x)(1)}{x^2} = \frac{\cancel{x} + x\sec x\tan x - \cancel{x} - \sec x}{x^2}$

$g'(x) = \frac{\sec x(x\tan x - 1)}{x^2}$

$g'(2\pi) = \frac{1(-1)}{(2\pi)^2} = \frac{-1}{(2\pi)^2}$

$g'(\pi) = \frac{(-1)(-1)}{(\pi)^2} = \frac{1}{(\pi)^2}$

$\frac{h(2\pi) - h(\pi)}{2\pi - \pi}$

$\frac{(2\pi)^2 g'(2\pi) - (\pi)^2 g'(\pi)}{\pi} = \frac{(2\pi)^2\left(\frac{-1}{(2\pi)^2}\right) - (\pi)^2\left(\frac{1}{(\pi)^2}\right)}{\pi} = \frac{-1-1}{\pi} = -\frac{2}{\pi}$

(A) -2 (B) 2 (C) $-\frac{2}{\pi}$ (D) $\frac{2}{\pi}$ (E) 0

B 9. If $\frac{d}{dx}\left[\left(\frac{x+3}{2x-1}\right)^4\right] = 4\left(\frac{x+3}{2x-1}\right)^3\left(\frac{(2x-1)(1)-(x+3)(2)}{(2x-1)^2}\right) = \frac{4(x+3)^3\left[\cancel{2x}-1-\cancel{2x}-6\right]}{(2x-1)^5}$

$= -28\frac{(x+3)^3}{(2x-1)^5}$

(A) $-28\frac{(x+3)^3}{(2x-1)^3}$ (B) $-28\frac{(x+3)^3}{(2x-1)^5}$ (C) $28\frac{(x+3)^3}{(2x-1)^5}$ (D) $-20\frac{(x+3)^3}{(2x-1)^3}$ (E) $20\frac{(x+3)^3}{(2x-1)^5}$

B 10. If $f(1) = \frac{\pi}{4}$ and $f'(1) = 3$, find the equation of the tangent line to $h(x) = \cot(f(x))$ at $x = 1$.

(A) $6x - y = 7$ (B) $6x + y = 7$ (C) $6x - y = -7$ (D) $-6x + y = 2$ (E) $6x - y = 7$

$h'(x) = -\csc^2(f(x))\cdot f'(x)$, $h'(1) = -f'(1)\left[\csc(f(1))\right]^2$
$= (-3)(\sqrt{2})^2 = -6 = m$

$h(1) = \cot(f(1)) = \cot\left(\frac{\pi}{4}\right) = 1$, pt: $(1,1)$

$y = 1 - 6(x-1)$

$y = 1 - 6x + 6$

$6x + y = 7$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	-3	2	-1	$\frac{1}{2}$
2	1	-2	3	$\frac{1}{4}$
3	5	$4a$	0	$\frac{1}{6}$

$(4a)\left(\frac{1}{4}\right) = \frac{-1}{\left(\frac{1}{2}\right)(-2)}$

$a = \frac{-1}{-1}$

$a = 1$

B 11. The table above gives the values for differentiable functions $f(x)$ and $g(x)$ & their derivatives at selected values. Find the value of a (if it exists) so that the tangent lines to $f(g(x))$ and $g(f(x))$ are perpendicular at $x = 2$.

$\frac{d}{dx}[f(g(x))]$ $\frac{d}{dx}[g(f(x))]$

$m_1 = f'(g(x))\cdot g'(x)$ $g'(f(x))\cdot f'(x) = m_2$

(A) $-\frac{1}{2}$ (B) 1 (C) 2 (D) -5 (E) DNE

So, $m_1 = -\frac{1}{m_2}$ (perpendicular at $x=2$)

$f'(g(2))\cdot g'(2) = \frac{-1}{g'(f(2))\cdot f'(2)}$

$f'(3)\cdot\left(\frac{1}{4}\right) = \frac{-1}{g'(1)\cdot(-2)}$

C 12. $\frac{d^{50}}{dx^{50}}[\sin 3x] =$

$4\overline{)50}$ 12, 48, 2 ← Remainder

So $\frac{d^{50}y}{dx^{50}}[\sin x] = \frac{d^2y}{dx^2}[\sin x] = -\sin x$

(A) $3^{50}\sin 3x$ (B) $3^{50}\cos 3x$ (C) $-3^{50}\sin 3x$ (D) $-3^{50}\cos 3x$ (E) $150\cos 3x$

$y^{0} = \sin 3x$

$y' = 3\cos 3x$

$y'' = -3^2\sin 3x$

$y''' = -3^3\cos 3x$

$y^{(4)} = 3^4\sin 3x$

repeats every 4, with 3^n in front.

So $\frac{d^{50}}{dx^{50}}[\sin 3x] = 3^{50}(-\sin(3x)) = -3^{50}\sin(3x)$

A 13. Find the slope of the tangent line to $x^2y^2 = (x+2)^2(40-y^2)$ at $(-3,2)$.

$\frac{d}{dx}[x^2y^2] = \frac{d}{dx}[(x+2)^2(40-y^2)]$

$2xy^2 + x^2\cdot 2y\frac{dy}{dx} = 2(x+2)^1(40-y^2) + (x+2)^2\left(-2y\frac{dy}{dx}\right)$

At $(-3,2)$: $2(-3)(4) + (9)(2)(2)\frac{dy}{dx} = 2(-1)(40-4) + (-1)^2\left(-4\frac{dy}{dx}\right)$

$-24 + 36\frac{dy}{dx} = -72 - 4\frac{dy}{dx}$

$40\frac{dy}{dx} = -48$, $\frac{dy}{dx} = \frac{-48}{40} = -\frac{6}{5}$

(A) $-\frac{6}{5}$ (B) $\frac{3}{5}$ (C) $\frac{9}{10}$ (D) $-\frac{3}{10}$ (E) 4

D 14. If $\cos(xy) = \frac{\sqrt{3}}{2}$, find $\frac{d^2y}{dx^2}$.

(A) $\frac{1}{\sin(xy)}$ (B) $-\frac{1}{\sin(xy)}$ (C) 0 (D) $\frac{2y}{x^2}$ (E) $-\frac{2y}{x^2}$

$\frac{d}{dx}[\cos(xy)] = \frac{d}{dx}\left[\frac{\sqrt{3}}{2}\right]$

$-\sin(xy)\left(1\cdot y + x\cdot\frac{dy}{dx}\right) = 0$

*don't divide out variables with zero

$-y\sin(xy) = x\frac{dy}{dx}\sin(xy)$

$\boxed{\frac{dy}{dx} = -\frac{y}{x}}$

$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[-\frac{y}{x}\right]$

$\frac{d^2y}{dx^2} = \frac{(x)\left(-\frac{dy}{dx}\right) - (-y)(1)}{x^2}$

$\frac{d^2y}{dx^2} = \frac{-x\left(-\frac{y}{x}\right) + y}{x^2}$

$\frac{d^2y}{dx^2} = \frac{2y}{x^2}$

A 15. When the gates opened at the Unicorn football game, everyone rushed in. The rate R at which rabid fans entered Unicorn stadium at different times is given in the table below. Based on the data below, which of the following statements is accurate?

time (sec)	0	25	50	75	100	180
R (people/sec)	70	120	220	340	300	160

✓ I. At $t = 2$ seconds, the instantaneous rate of change of R is approximately 2 people/sec^2

X II. At $t = 24$ seconds, the instantaneous rate of change of R is approximately 8 people/sec^2

X III. At $t = 75$ seconds, more people are entering Unicorn stadium that at any other time during the first 1.5 minutes.

(A) I only (B) II only (C) I and III only (D) I and II only (E) I, II, and III

I. $R'(2) \approx \frac{120-70}{25-0} = \frac{50}{25} = 2$ ppl/sec^2

II. $R'(24) \approx \frac{120-70}{25-0} = \frac{50}{25} = 2$ ppl/sec^2 $\neq 8$ ppl/sec^2

III. The rate at $t=75$ sec is the largest rate in the table, but there is data in between the data points given, about which we know nothing. Who's to say at $t=76$ sec, people weren't rushing it at 350 ppl/sec? Certainly not I.

Part II: Free Response—Show all set ups, use correct notation, indicate your methods, and answer in complete math/English sentences (with units) when appropriate.

16. An elephant moves along the x-axis so that at any time $t \in [0, 2\pi]$ seconds, its position, in feet, is given by $x(t) = 2t\sin t + 2\cos t + t^2$.

(a) Determine if the speed of the elephant is increasing or decreasing at $t = \frac{\pi}{6}$ seconds. Justify your answer.

$x'(t) = v(t) = \cancel{2\sin t} + 2t\cos t - \cancel{2\sin t} + 2t$

$v(t) = 2t\cos t + 2t$

$v\left(\frac{\pi}{6}\right) = (2)\left(\frac{\pi}{6}\right)\left(\cos\frac{\pi}{6}\right) + 2\left(\frac{\pi}{6}\right)$ $\sqrt{1}$

$= \left(\frac{\pi}{3}\right)\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{3} > 0$

$x''(t) = v'(t) = a(t) = 2\cos t - 2t\sin t + 2$

$a\left(\frac{\pi}{6}\right) = (2)\left(\cos\frac{\pi}{6}\right) - (2)\left(\frac{\pi}{6}\right)\left(\sin\frac{\pi}{6}\right) + 2$

$= (2)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\pi}{3}\right)\left(\frac{1}{2}\right) + 2$

$= \sqrt{3} - \frac{\pi}{6} + 2$ $\sqrt{2}$

$= 2 + \sqrt{3} - \frac{\pi}{6} > 0$

Since $v\left(\frac{\pi}{6}\right) > 0$ and $a\left(\frac{\pi}{6}\right) > 0$, $\sqrt{4}$ the speed of the elephant is increasing at $t = \frac{\pi}{6}$ seconds. $\sqrt{3}$

(b) For what values of $t \in (0, 2\pi)$ is the elephant at rest? Show the work that leads to your answer.

$v(t) = 0$

$2t\cos t + 2t = 0$

$2t(\cos t + 1) = 0$

$2t = 0$, $\cos t + 1 = 0$

$t = 0$ sec, $\cos t = -1$

$t = 0 \notin (0, 2\pi)$ $\quad t = \pi$ sec

So the elephant is at rest at $t = \pi$ seconds. $\sqrt{5}$

(c) On the interval $0 \le t \le 2\pi$ seconds, how far does the elephant travel? Justify your answer.

$v(t) = 2t(\cos t + 1)$

Since $2t \geq 0$ and $\cos t + 1 \geq 0$ for $0 \le t \le 2\pi$

$v(t) \geq 0$ for all $t \in [0, 2\pi]$ $\sqrt{6}$

So the elephant is <u>NOT</u> moving left at any time. This mean's the pachyderm's displacement is <u>also</u> its distance travelled.

$x(t) = 2t\sin t + 2\cos t + t^2$

$*$ Displacement $= x(2\pi) - x(0)$ $\sqrt{7}$

$= \left(4\pi\sin 2\pi + 2\cos 2\pi + (2\pi)^2\right) - \left(2(0)\sin 0 + 2\cos 0 + 0^2\right)$

$= 0 + 2 + 4\pi^2 - 0 - 2 - 0$

$= 4\pi^2$ ft $\sqrt{8}$

units on (b) & (c) $\sqrt{9}$
seconds ft