Name KEY ______ Date ______ Santa's Hat Size ______

AP Calculus TEST: 2.1 - 2.10, NO CALCULATOR

Part One: Multiple Choice—Put the correct CAPITAL letter in the space to the left of each question.

D 1. If $f(x)=\sec^{-1}(x)$, what is $\lim\limits_{h\to 0}\frac{f(-2+h)-f(-2)}{h}$? $= f'(-2)$ for $f(x)=\sec^{-1}x = \text{arcsec}\,x$

(A) DNE (B) $\frac{-1}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{3}}$ (D) $\frac{1}{2\sqrt{3}}$ (E) $\frac{-1}{2\sqrt{3}}$

$f' = \frac{1}{|x|\sqrt{x^2-1}}$

$f'(-2) = \frac{1}{|-2|\sqrt{(-2)^2-1}}$

$= \frac{1}{2\sqrt{3}}$

B 2. $\frac{d}{dx}\left[(\sin x)^x\right]=$ (A) $\ln(\sin x)+x\cot x$ (B) $(\ln(\sin x)+x\cot x)(\sin x)^x$

(C) $(\ln(\sin x)+\cot x)(\sin x)^x$ (D) $x(\sin x)^{x-1}$ (E) $\ln(\sin x)\cdot\cos x\cdot(\sin x)^x$

Log Diff $y=(\sin x)^x$

$\ln y = \ln(\sin x)^x$

$\ln y = x\cdot\ln(\sin x)$

$\frac{d}{dx}: \frac{1}{y}\cdot\frac{dy}{dx} = 1\cdot\ln(\sin x) + x\cdot\left(\frac{1}{\sin x}\right)(\cos x)$

$\frac{dy}{dx} = \left[\ln(\sin x) + x\cdot\cot x\right]\cdot y$

$\frac{dy}{dx} = \left[\ln(\sin x) + x\cot x\right](\sin x)^x$

A 3. $\frac{d}{dx}\left[5^{2x}+\log_5(2x)\right]=$

(A) $\ln 25\cdot 5^{2x}+\frac{1}{x\ln 5}$ (B) $\ln 5\cdot 5^{2x}+\frac{1}{x\ln 5}$ (C) $\ln 25\cdot 5^{2x}+\frac{2}{x\ln 5}$

(D) $\ln 5\cdot 5^{2x}+\frac{1}{2x\ln 5}$ (E) $\ln 25\cdot 5^{2x}+\frac{1}{2x\ln 5}$

$= 5^{2x}\cdot\ln 5\cdot 2 + \frac{1}{(2x)\ln 5}\cdot(2)$

$= 2\ln 5\cdot 5^{2x} + \frac{\cancel{2}}{\cancel{2}\cdot x\cdot\ln 5}$

$= \ln 25\cdot 5^{2x} + \frac{1}{x\ln 5}$

A 4. What is the slope of the graph of $xy^2+x^2y=2$ at the point $(1,1)$?

(A) -1 (B) 1 (C) $\frac{2}{3}$ (D) $\frac{-2}{3}$ (E) 0

$\frac{d}{dx}: 1\cdot y^2 + x\cdot 2y\frac{dy}{dx} + 2xy + x^2\cdot\frac{dy}{dx} = 0$

At $(1,1)$: $1 + 2\frac{dy}{dx} + 2 + \frac{dy}{dx} = 0$

$3+3\frac{dy}{dx}=0$

$3\frac{dy}{dx}=-3$

$\frac{dy}{dx}=-1$

B ____ 5. If $f(x) = \ln\left(x + 4 + e^{-3x}\right)$, then $f'(0)$ is

(A) $\frac{2}{5}$ (B) $\frac{-2}{5}$ (C) $\frac{1}{5}$ (D) $\frac{1}{4}$ (E) DNE

$f'(x) = \left(\frac{1}{x+4+e^{-3x}}\right)\left(1+0+e^{-3x}(-3)\right)$

$f'(x) = \frac{1-3e^{-3x}}{x+4+e^{-3x}}$

$f'(0) = \frac{1-3e^{0}}{0+4+e^{0}}$

$f'(0) = \frac{1-3}{4+1}$

$f'(0) = \frac{-2}{5}$

E ____ 6. If $f(x) = 2\sin x \cos x$, what is $f'(x)$?

(A) $-2\cos x \sin x$ (B) $-2\cos^2 x$ (C) $-2\sin^2 x$ (D) $\cos(2x)$ (E) $2\cos(2x)$

Note: $2\sin x\cos x = \sin 2x$ (Double-Angle Identity)

so $f(x) = \sin(2x)$

$f'(x) = \cos(2x) \cdot 2$

$f'(x) = 2\cos(2x)$

or $f(x) = (2\sin x)(\cos x)$

$f'(x) = 2\cos x \cdot \cos x + 2\sin x(-\sin x)$

$f'(x) = 2\cos^2 x - 2\sin^2 x$

$f'(x) = 2(\cos^2 x - \sin^2 x)$

$f'(x) = 2\cos 2x$

Note: $\cos^2 x - \sin^2 x = \cos 2x$ (Double-Angle Identity)

A ____ 7. At what value of x does the graph of $f(x) = \frac{e^x}{x+1}$ have a horizontal tangent?

(A) $x = 0$ (B) $x = 1$ (C) $x = -1$ (D) $x = e$ (E) at no x-value

$f'(x) = \frac{(x+1)(e^x) - (e^x)(1)}{(x+1)^2}$

$f'(x) = \frac{e^x\left[(x+1) - 1\right]}{(x+1)^2}$

$f'(x) = \frac{e^x \cdot x}{(x+1)^2} = 0$

when $e^x \cdot x = 0$

$e^x = 0$ (Never) or $\boxed{x = 0}$

C ____ 8. $\frac{d}{dx}\left[e^{\ln(\ln x)}\right] =$ (A) $\ln x$ (B) x (C) $\frac{1}{x}$ (D) e^x (E) $e^{\ln(\ln x)}$

$e^{\ln x} = x$

so $\frac{d}{dx}\left[e^{\ln(\ln x)}\right]$

$= \frac{d}{dx}\left[\ln x\right] = \frac{1}{x}$

B ____ 9. If $y = \ln\left[\frac{2x^2}{\sqrt{x+3}}\right]$, what is $\left.\frac{dy}{dx}\right|_{x=1}$?

(A) 0 (B) $\frac{15}{8}$ (C) $\frac{7}{4}$ (D) $\ln\left(\frac{15}{8}\right)$ (E) $\ln\left(\frac{7}{4}\right)$

Simplify early & often, especially when you have logs.

$y = \ln 2 + 2\ln x - \frac{1}{2}\ln(x+3)$

$\frac{dy}{dx} = 0 + 2\left(\frac{1}{x}\right) - \frac{1}{2}\left(\frac{1}{x+3}\right)(1)$

$\left.\frac{dy}{dx}\right|_{x=1} = \frac{2}{1} - \frac{1}{2}\left(\frac{1}{1+3}\right)$

$= 2 - \frac{1}{2(4)}$

$= 2 - \frac{1}{8} = \boxed{\frac{15}{8}}$

Part Two: Free Response

10. Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

(a) Write an equation for the line tangent to the curve at the point $(-1,1)$.

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{1}{3(1^2)-(-1)} = \frac{1}{4} \quad \checkmark 1$$

So equation is

$$y = 1 + \frac{1}{4}(x+1) \quad \checkmark 2$$

(b) Find the coordinates (x, y) of all the points on the curve at which the line tangent to the curve at that point is vertical.

$$\frac{dy}{dx} = \begin{cases} \text{DNE} \\ \text{or} \\ \pm\infty \\ \text{or} \\ \frac{\neq 0}{0} \end{cases}$$

$$3y^2 - x = 0 \quad \checkmark 3$$

$$x = 3y^2$$

So, $y^3 - xy = 2$ becomes

$$y^3 - (3y^2)y = 2 \quad \checkmark 4$$

$$y^3 - 3y^3 = 2$$

$$-2y^3 = 2$$

$$y^3 = -1$$

$$y = -1$$

when $y=-1$:

$x = 3(-1)^2$ or $(-1)^3 - x(-1) = 2$

$x = 3$ or $-1 + x = 2$

$x = 3$

So the point of vertical tangency is $(3,-1)$ $\checkmark 5$

(c) Evaluate $\frac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

$$\frac{dy}{dx} = \frac{y}{3y^2 - x}$$

$$\frac{d}{dx}: \quad \frac{d^2y}{dx^2} = \frac{(3y^2-x)\frac{dy}{dx} - (y)\left(6y\frac{dy}{dx} - 1\right)}{(3y^2-x)^2}$$

Implicit Diff $\checkmark 6$ & $\checkmark 7$

at $(-1,1)$, $\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{1}{4}$ (part (a))

$\checkmark 8$

$$\text{so } \left.\frac{d^2y}{dx^2}\right|_{(-1,1)} = \frac{(3-(-1))\left(\frac{1}{4}\right) - (1)\left(6(1)\left(\frac{1}{4}\right) - 1\right)}{(3(1)^2 - (-1))^2}$$

$\checkmark 9$ final numeric answer

$$= \frac{4\left(\frac{1}{4}\right) - \left(\frac{3}{2} - 1\right)}{(4)^2} = \frac{1 - \frac{1}{2}}{16} = \frac{\frac{1}{2}}{16} = \frac{1}{32}$$