Name Key Date______ Period______

Worksheet 3.2—Rolle's Theorem and the MVT

Show all work. No calculator unless otherwise stated.

Multiple Choice

special case of MVT where $f(b)=f(a)$.

D 1. Determine if the function $f(x)=x\sqrt{6-x}$ satisfies the hypothesis of Rolle's Theorem on the interval $[0,6]$, and if it does, find all numbers c satisfying the conclusion of that theorem.

(A) 2, 3 (B) 4, 5 (C) 5 (D) 4 (E) hypothesis not satisfied

$f(x)$ cont $\forall x \in [0,6]$.
$f(x)$ diff'able $\forall x \in (0,6)$.
$f(0)=0=f(6)$

$f(x)=x(6-x)^{1/2}$
$f'(x)=(6-x)^{1/2}+\frac{1}{2}x(6-x)^{-1/2}(-1)$
$f'(x)=0$
$\frac{1}{2}(6-x)^{-\frac{1}{2}}\left[2(6-x)^1-x\right]=0$
$\frac{12-3x}{2\sqrt{6-x}}=0$
$12-3x=0$
$x=4$

E 2. Let f be a function defined on $[-1,1]$ such that $f(-1)=f(1)$. Consider the following properties that f might have:

I. f is continuous on $[-1,1]$, differentiable on $(-1,1)$.

II. $f(x)=\cos^3 x$ $=(\cos x)^3$ $f'(x)=3\cos^2 x(-\sin x)$ cont & diff'able

III. $f(x)=|\sin \pi x|$ $f(x)=\sqrt{(\sin\pi x)^2}$ $f'(x)=\frac{1}{2}(\sin^2\pi x)^{-\frac{1}{2}}\cdot 2\sin\pi x\cdot\cos\pi x\cdot\pi=\frac{\sin\pi x\cdot\cos\pi x}{|\sin\pi x|}$

Not diff'able when $\sin\pi x=0$
$\pi x=0+\pi n$
$x=0+n, n\in\mathbb{Z}$
Not diffable on integers but IS diffable $\forall x\in(-1,1)$

Which properties ensure that there exists a c in $(-1,1)$ at which $f'(c)=0$?

(A) I only (B) I and II only (C) I and III only (D) II and III only (E) I, II, and III

D 3. Determine if the function $f(x)=x^3-x-1$ satisfies the hypothesis of the MVT on $[-1,2]$. If it does, find all possible values of c satisfying the conclusion of the MVT.

(A) $-\frac{1}{2}$

(B) $-1,\ 1$

(C) 0

(D) 1

(E) hypothesis not satisfied

f cont. $\forall x\in[-1,2]$.
f diff'able $\forall x\in(-1,2)$.

$f'(x)=\frac{f(2)-f(-1)}{2--1}$
$3x^2-1=\frac{5--1}{3}$
$3x^2-1=2$
$3x^2=3$
$x=\pm 1$ $x=-1\notin(-1,1)$!
$x=1$

cusp! $x=0$ cusp! $x=1$

A 4. Determine if the function $f(x)=x+x^{2/3}(1-x)^{1/3}$ satisfies the hypothesis of the MVT on $[0,1]$. If it does, find all possible values of c satisfying the conclusion of the MVT. (You will have to factor out least powers.)

(A) $\frac{2}{3}$

(B) $\frac{1}{4}$

(C) $\frac{1}{2}$

(D) $\frac{1}{3}$

(E) hypothesis not satisfied

f cont. $\forall x \in [0,1]$
f diff'able $\forall x \in (0,1)$ open!

$$f'(x) = \frac{f(1)-f(0)}{1-0}$$

$$1+\frac{2}{3}x^{-\frac{1}{3}}(1-x)^{1/3}+x^{\frac{2}{3}}\left(\frac{1}{3}(1-x)^{-\frac{2}{3}}(-1)\right)=1$$

$$\frac{2}{3}x^{-\frac{1}{3}}(1-x)^{1/3}-\frac{1}{3}x^{\frac{2}{3}}(1-x)^{-\frac{2}{3}}=0$$

$$\frac{1}{3}x^{-\frac{1}{3}}(1-x)^{-\frac{2}{3}}\left[2(1-x)^{1}-x\right]=0$$

$$\frac{1}{3}x^{-\frac{1}{3}}(1-x)^{-\frac{2}{3}}\left[2-3x\right]=0$$

$x=0$ $x=1$ $\boxed{x=\frac{2}{3}}$

Remember $c\in(0,1)$!

B 5. Which of the following functions below satisfy the hypothesis of the MVT?

I. $f(x)=\dfrac{1}{x+1}$ on $[0,2]$ $x\neq -1$

II. $f(x)=x^{1/3}$ on $[0,1]$ cusp @ $x=0$ but still diff'able $\forall x\in(0,1)$

III. $f(x)=|x|$ on $[-1,1]$ not diff'able @ $x=0 \in (-1,1)$

(A) I only (B) I and II only (C) I and III only (D) II only (E) II and III only

C 6. As a graduation present, Jenna received a sports car which she drives very fast but very, very smoothly and safely. She always covers the 53 miles from her apartment in Austin, Texas to her parents' home in New Braunfels in less than 48 minutes. To slow her down, her dad decides to change the speed limit (he has connections.) Which one of the speed limits below is the highest speed her father can post, but still catch her speeding at some point on her trip?

(A) 55 mph (B) 70 mph (C) 65 mph (D) 50 mph (E) 60 mph

$$\frac{53\text{ miles}}{48\text{ min}}\times\frac{60\text{ min}}{1\text{ hr}}=66.25\text{ mph}.$$

B 7. Consider the following statements:

I. $f(x)$ is continuous on $[a,b]$

II. $f(x)$ is differentiable on (a,b)

III. $f(a)=f(b)$ (special case!)

Which of the above statements are required in order to guarantee a $c\in(a,b)$ such that $f'(c)(b-a)=f(b)-f(a)$?

(A) I only (B) I and II only (C) I, II, and III (D) III only (E) I and III only

Short Answer

8. Without looking at your notes, state the Mean Value Theorem.

If . . . $f(x)$ is continuous $\forall x \in [a,b]$ and diff'able $\forall x \in (a,b)$

then . . . $\exists\ c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

9. Determine if Rolle's Theorem can be applied to the following functions on the given interval. If so, find the value(s) guaranteed by the theorem.

(a) $f(x) = \cos 2x$ on $\left[-\dfrac{\pi}{12}, \dfrac{\pi}{6}\right]$

f is cont. $\forall x \in \left[-\frac{\pi}{12}, \frac{\pi}{6}\right]$.

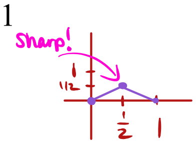
f is diff'able $\forall x \in \left(-\frac{\pi}{12}, \frac{\pi}{6}\right)$.

$f\left(-\frac{\pi}{12}\right) = \cos\left(2\left(-\frac{\pi}{12}\right)\right) = \cos\left(-\frac{\pi}{6}\right) = \cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$

$f\left(\frac{\pi}{6}\right) = \cos\frac{2\pi}{6} = \cos\frac{\pi}{3} = \frac{1}{2}$

Since $f\left(-\frac{\pi}{12}\right) \neq f\left(\frac{\pi}{6}\right)$, Rolle's Thm does not apply.

(b) $g(x) = \begin{cases} x, & 0 \le x \le \dfrac{1}{2} \\ 1-x, & \dfrac{1}{2} < x \le 1 \end{cases}$ on $[0,1]$



g is cont $\forall x \in [0,1]$.

Since g is not diff'able at $x = \frac{1}{2} \in (0,1)$, Rolle's Thm does not apply.

10. Determine if the MVT can be applied to the following functions on the given interval. If so, find the exact value(s) guaranteed by the theorem. Be sure to show your set up in finding the value(s).

(a) $f(x) = \ln(x-1)$ on $[2,4]$

f cont. $\forall x \in [2,4]$.

f diff'able $\forall x \in (2,4)$.

$f'(x) = \dfrac{f(4)-f(2)}{4-2}$

$\dfrac{1}{x-1} = \dfrac{\ln 3 - \ln 1}{2}$

$\dfrac{1}{x-1} = \frac{1}{2}\ln 3$

$\dfrac{2}{\ln 3} = x-1$

$\dfrac{2}{\ln 3} + 1 = x$

(b) $f(x) = \begin{cases} \arcsin x, & -1 \le x < 1 \\ \dfrac{x}{2}, & 1 \le x \le 3 \end{cases}$ on $[-1,3]$.

Since f is not cont. at $x = 1 \in [-1,3]$, MVT does not apply.

(c) $g(x) = \dfrac{x+1}{x}$ on $\left[\dfrac{1}{2}, 2\right]$

$g(x) = 1 + x^{-1}$

g is cont. $\forall x \in \left[\frac{1}{2}, 2\right]$.

g is diff'able $\forall x \in \left(\frac{1}{2}, 2\right)$.

$g'(x) = \dfrac{g(2) - g\left(\frac{1}{2}\right)}{2 - \frac{1}{2}}$

$-x^{-2} = \dfrac{\frac{3}{2} - 3}{\frac{3}{2}}\left(\frac{2}{2}\right)$

$-\dfrac{1}{x^2} = \dfrac{3-6}{3}$

$-\dfrac{1}{x^2} = -1$

$1 = x^2 \quad x = \pm 1$

$\boxed{x = 1}$ $\quad x = -1 \notin \left[\frac{1}{2}, 2\right]$

(d) $f(x) = 2\sin x + \sin 2x$ on $[0, \pi]$

f is cont $\forall x \in [0,\pi]$

f is diff'able $\forall x \in (0,\pi)$.

$f'(x) = \dfrac{f(\pi) - f(0)}{\pi - 0}$

$2\cos x + 2\cos 2x = 0$

$2\cos x + 2(\cos^2 x - \sin^2 x) = 0$

$2\cos x + 2\cos^2 x - 2\sin^2 x = 0$

$2\cos x + 2\cos^2 x - 2(1 - \cos^2 x) = 0$

$4\cos^2 x + 2\cos x - 2 = 0$

$2(2\cos^2 x + \cos x - 1) = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

$2\cos x - 1 = 0 \qquad \cos x + 1 = 0$

$\cos x = \frac{1}{2} \qquad \cos x = -1$

$x = \frac{\pi}{3} \qquad x = \pi \notin (0, \pi)$

11. (**Calculator permitted**) For $f(x) = -x^4 + 4x^3 + 8x^2 + 5$ on $[0,5]$

(a) Determine if the MVT can be applied on the given interval. If so, find the value(s) guaranteed by the theorem.

f cont. $\forall x \in [0,5]$

f diff'able $\forall x \in (0,5)$.

$f'(x) = \frac{f(5)-f(0)}{5-0}$

$f'(x) = 15$

$f'(x) - 15 = 0 \leftarrow$ in Y_3 $Y_4 = 0$ find inters.

$x = 0.673$ $(0.674) = A$

$x = 3.793$ $(3.794) = B$

(b) Find the equation of the secant line on $[0,5]$

$y = 5 + 15(x - 0)$

(c) Find the equation of the tangent line at any value of c found above.

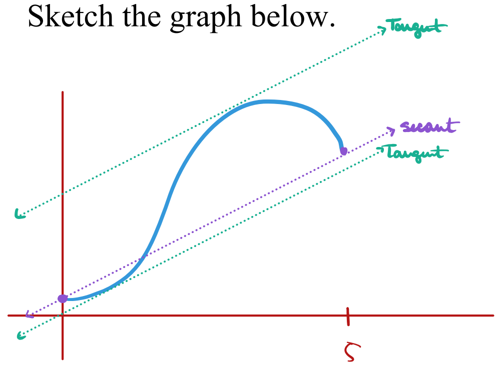
$m = 15$

Eq 1: $y = f(A) + 15(x - A)$ eq 2: $y = f(B) + 15(x - B)$

$f(A) = 9.646$ (9.647) $f(B) = 131.402$ (131.403)

Tang eq 1: $y = 9.646 + 15(x - 0.673)$ Tang eq 2: $y = 131.402 + 15(x - 3.793)$

(d) On your calculator, sketch a graph of $f(x)$ on $[0,5]$ along with the secant and tangent line(s). Sketch the graph below.



12. Let f satisfy the hypothesis of Rolle's Theorem on an interval $[a,b]$, such that $f'(c) = 0$. Using your knowledge of transformations, find an interval, in terms of a and b, for the function g over which Rolle's theorem can be applied, and find the corresponding critical value of g, in terms of c. Assume k is a non-zero constant such that $k > 0$.

(a) $g(x) = f(x) + k$ vert. shift k units up. (vertical has no change to x-vals)

New Interval: $[a,b]$

New x-value: $f'(c) = 0$

(b) $g(x) = f(x - k)$ horiz. shift right k units

New Interval: $[a+k, b+k]$

New x-value: $f'(c+k) = 0$

(c) $g(x) = kf(x)$ vert. dilation $0<k<1$ comp $k>1$ stretch

New Interval: $[a,b]$

New x-value: $f'(c) = 0$

(d) $g(x) = f(kx)$ horiz. dilation (depends on $0<k<1$ str $k>1$ compr.)

New Interval: $[\frac{a}{k}, \frac{b}{k}]$

New x-value: $f'(\frac{c}{k}) = 0$

13. The function $f(x)=\begin{cases}0, & x=0\\ 1-x, & 0<x\le 1\end{cases}$ is differentiable on $(0,1)$ and satisfies $f(0)=f(1)$. However, its derivative is never zero on $(0,1)$. Does this contradict the Mean Value Theorem? Explain why or why not.

The fxn is not cont. at $x=0 \in [0,1]$,
therefore MVT does not apply.

14. Determine the values of *a*, *b*, and *c* such that the function *f* satisfies the hypothesis of the MVT on the interval $[0,3]$.

$$f(x)=\begin{cases}1, & x=0\\ ax+b, & 0<x\le 1\\ x^2+4x+c, & 1<x\le 3\end{cases}$$

Continuity

f must be cont. $\forall x \in [0,3]$

So $ax+b=1$ when $x=0$, $b=1$

$ax+b=x^2+4x+c$ when $x=1$

$a+1=5+c$

$a=4+c$

Diff'ability

f must be diff'able $\forall x \in (0,3)$ (check at $x=1$)

$f'(x)=\begin{cases}a & x\le 1\\ 2x+4 & x>1\end{cases}$

$\lim_{x\to 1^-} f'(x)=\lim_{x\to 1^+} f'(x)$

$a=2(1)+4$

$a=6$

$a=4+c$

$6=4+c$

$c=2$

$a=6, b=1, c=2$.

15. Suppose that we know that $f(x)$ is continuous and differentiable on $[6,15]$. Let's also suppose that we know that $f(6)=-2$ and that $f'(x)\le 10$ for all $x\in[6,15]$. What is the largest possible value for $f(15)$?

$f'(x)\le 10$

$\dfrac{f(15)-f(6)}{15-6}\le 10$

$\dfrac{f(15)+2}{9}\le 10$

$f(15)+2\le 90$

$f(15)\le 88$

max value of $f(15)$ is 88.

16. Let $f(x)=\tan x$. Show that $f(\pi)=f(2\pi)$ but that there is not number $c\in(\pi,2\pi)$ such that $f'(c)=0$. Why does this not contradict Rolle's Theorem?

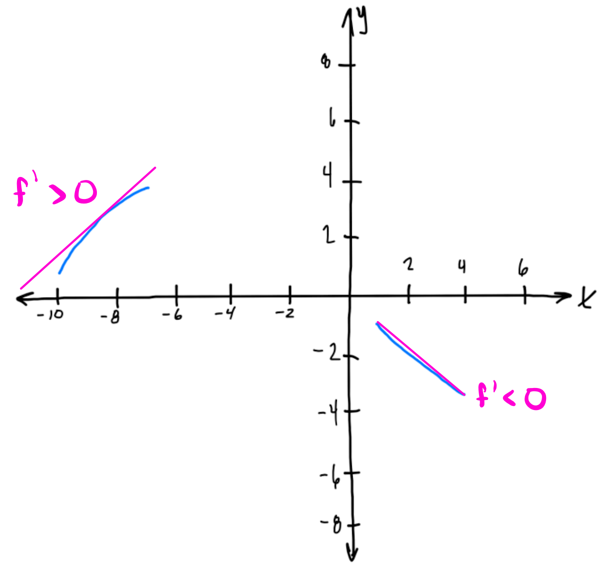
$f(\pi)=\tan\pi=0$

$f(2\pi)=\tan 2\pi=0$

f is not cont @ $x=\frac{3\pi}{2}\in[\pi,2\pi]$

Rolle's Thrm does not apply.

17. The figure at right shows two **parts** of the graph of a function $f(x)$ that is continuous on $[-10,4]$ and differentiable on $(-10,4)$. It so happens that the derivative $f'(x)$ is also continuous on $[-10,4]$.



(a) Explain why f must have at least one zero in $[-10,4]$.

$f(-10) > 0$ and $f(4) < 0$

Since f is cont. $\forall x \in [-10,4]$ & diff'able $\forall x \in (-10,4)$ and $f(4) < 0 < f(-10)$

$\exists\ c \in (-10,4) \mid f(c) = 0$

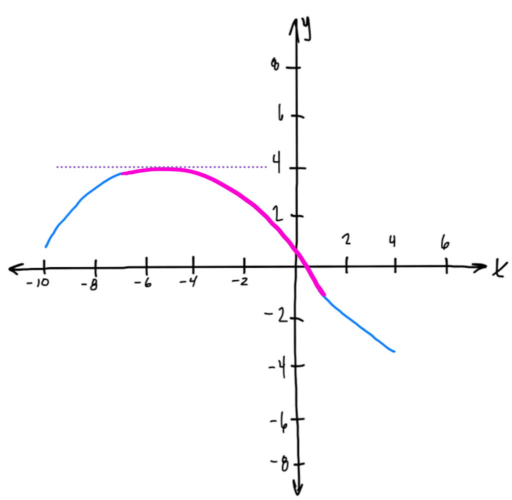
(b) Explain why f' must also have at least one zero in the interval $[-10,4]$. What are these zeros called?

Since $f'(3) < 0 < f'(-9)$ and f' is cont $\forall x \in [-10,4]$

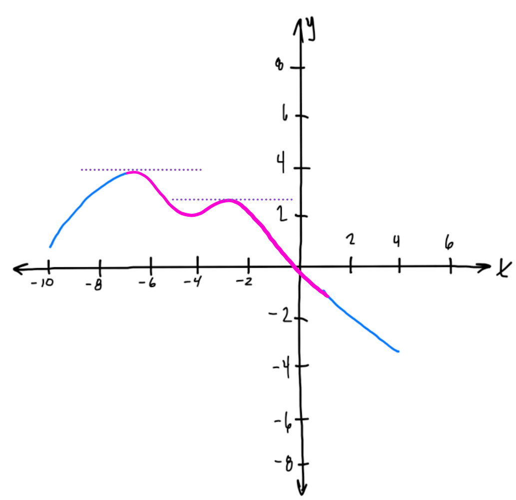
$\exists\ c \in [-9,3] \mid f'(c) = 0$.

critical points!

(c) Make a possible sketch of the function with **one** zero of f' on the interval $[-10,4]$.



(d) Make a **possible** sketch of the function with **at least two** zeros of f' on the interval $[-10,4]$.



(e) Were the conditions of continuity of f and f' necessary to do parts (a) through (d)? Explain.

for (a) & (b) yes, b/c continuity

for (c) & (d) no, you could sketch a graph w/ one or two horiz. tangents that is not continuous.