

§12.1—Epsilon-Delta Proofs

When we mathematically state $\lim_{x \rightarrow 2} f(x) = 5$, we say “as x approaches 2 from both sides, y approaches 5.”

This is a very intuitive idea, but here are a couple of questions you might have asked yourself back in your early days of calculus:

- (1) Just HOW close to 5 must we get in order to say, confidently, that we are actually approaching 5 and not, let's say, 4.99?
- (2) And HOW close do we have to get to 2 from either side in order to make (1) happen?

That is, how can we PROVE that the limit as x approaches 2 is 5?

While Leibniz's method relied on a ratio of “infinitesimals,” Mr. Newton's method of fluxions (remember, he effectively divided by zero), used the notion of a limit when discussing his ratio $\frac{\dot{s}}{\dot{t}}$. From the *Principia* (1687), here are his words:

... the ultimate ratio of vanishing quantities is to be understood not as the ratio of quantities before they vanish or after they have vanished, but the ratio with which they vanish. . . There exists a limit which their velocity can attain at the end of the motion, but cannot exceed. This is their ultimate velocity. And it is the same for the limit of all quantities and proportions that come into being and cease existing. And since this limit is certain and definite, the determining of it is properly a geometrical problem.

So, Newton used the term “limit,” but never formalized its definition. Like anyone with a new toy, the 17th and 18th century mathematicians were too excited to explore the power of calculus and its consequences to give much attention to rigorously proving its theorems. Enter the 19th century and the age of mathematical analysis.



Following the work of Bohemian mathematician Bernard Bolzano (c. 1815), French mathematician Augustin-Louis Cauchy, (1821) stated his own definition of a limit:

When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the limit of all the others.

Cauchy began using the lower-case Greek letters “epsilon” (for French “erreur”) and “delta”, ϵ and δ in his limit arguments, which became the basis for the rigorously defining continuity, however he never gave a formal definition in terms of these variables. In wasn't until German mathematician Karl Weierstrass (c. 1838) that the formal definition of a limit, using epsilon and delta, was established. We shall get to that shortly.



To understand the need to formalize the definition of the limit, let's look at an example.

Example 1:

Let $f(x) = \begin{cases} 5x-2, & x \neq 1 \\ 4, & x = 1 \end{cases}$.

- (a) Using your intuition, as x gets close to 1, but not reaching 1, what values do $f(x)$ get close to, that is, what is $\lim_{x \rightarrow 1} f(x)$?

$\lim_{x \rightarrow 1} f(x) = 3$

- (b) How can we write “the distance from x to 1” and “the distance from $f(x)$ to 3” mathematically?

$|x-1|$

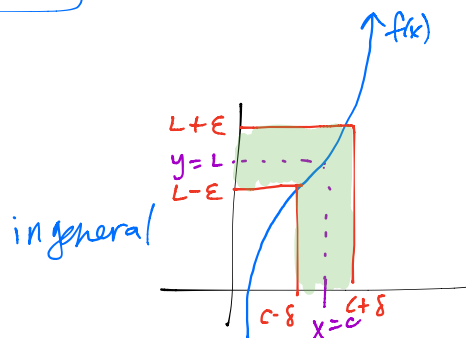
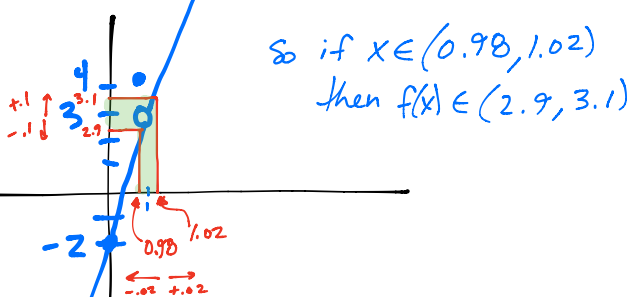
$|f(x)-3|$

- (c) How close does x have to be to 1 so that $f(x)$ differs from 3 by less than 0.1?

$|f(x)-3| < 0.1$ for some $|x-1| < \text{Distance}$

$|5x-2-3| < 0.1 \Rightarrow |5x-5| < 0.1$
 $5|x-1| < 0.1 \Rightarrow |x-1| < 0.02$
 $-0.02 < x-1 < 0.02$
 $0.98 < x < 1.02$

- (d) Interpret graphically your result from (c)



- (e) How close does x have to be to 1 so that $f(x)$ differs from 3 by less than 0.01?

$|5x-2-3| < 0.01$, for some $|x-1| < \text{Distance}$

$|5x-5|$
 $5|x-1| < 0.01$
 $|x-1| < 0.002$

So if $x \in (1-0.002, 1+0.002) = (0.998, 1.002)$
 then $f(x) \in (3-0.01, 3+0.01) = (2.99, 3.01)$

In the example above, 0.1 and 0.01 are called **error tolerances**, and we assign them the **epsilon value**, so $\varepsilon = 0.1$ and $\varepsilon = 0.01$. The answers to part (c) and (e) above we assign the value of **delta**, so $\delta = 0.02$ and $\delta = 0.002$ respectively. For the limit to exist, we must be able to bring $|f(x) - 3|$ within **ANY positive number ε** . This shouldn't be a problem.

In general, from Example 1, for any arbitrary ε , we can find a δ :

$$|f(x) - 3| < \varepsilon \text{ if } 0 < |x - 1| < \delta = \frac{\varepsilon}{5}$$

$\uparrow$ e.g. $x=1$

This can be rewritten to establish intervals for our $f(x)$ function and x :

$$3 - \varepsilon < f(x) < 3 + \varepsilon \text{ if } 1 - \delta < x < 1 + \delta$$

This means, in general, if we take a value of x from $(1 - \delta, 1 + \delta)$, our function value is guaranteed to fall within $(3 - \varepsilon, 3 + \varepsilon)$.

We are now ready for our formal definition similar to Weierstrass'.

$\varepsilon - \delta$ definition of a limit

If f is a function defined on an interval containing $x = c$, except possibly at $x = c$, then the limit of $f(x)$ as x approaches c is L . That is

$$\lim_{x \rightarrow c} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ when } 0 < |x - c| < \delta$$

or

$$\text{If } 0 < |x - c| < \delta, \text{ then } |f(x) - L| < \varepsilon$$

In essence, this means that function values of $f(x)$ can be made as close as we'd like to L simply by taking x close enough to c (without equaling c).

Alternatively, for each and every $\varepsilon > 0$, regardless of how small ε is, we can find a $\delta > 0$ such that $f(x)$ will lie in the open interval $(L - \varepsilon, L + \varepsilon)$ if x is chosen from the interval $(x - \delta, x + \delta)$.

A proof using the $\varepsilon - \delta$ definition of a limit is a two-step process: A guess, and a check.

Example 2:

Prove that $\lim_{x \rightarrow 1} (7x + 4) = 11$.

***Find δ as a function of ϵ**

Guess $|f(x) - L| < \epsilon$ for $|x - 1| < \delta$

$|7x + 4 - 11| < \epsilon$

$|7x - 7| < \epsilon$

$7|x - 1| < \epsilon$

$|x - 1| < \frac{\epsilon}{7}$

$\delta = \frac{\epsilon}{7}$

proof:

if $\delta = \frac{\epsilon}{7}$

then $|x - 1| < \delta$

$7|x - 1| < 7 \cdot \frac{\epsilon}{7} = \epsilon$

$|7x - 7| < \epsilon$

$|7x + 4 - 11| < \epsilon$

$|(7x + 4) - 11| < \epsilon$

$|f(x) - L| < \epsilon \checkmark$

Example 3:

(1969 BC-37) Of the following choices of δ , which is the largest that could be used successfully with an arbitrary ϵ in an epsilon-delta proof of $\lim_{x \rightarrow 2} (1 - 3x) = -5$?

(A) $\delta = 3\epsilon$

(B) $\delta = \epsilon$

(C) $\delta = \frac{\epsilon}{2}$

(D) $\delta = \frac{\epsilon}{4}$

(E) $\delta = \frac{\epsilon}{5}$

$|f(x) - L| < \epsilon$ for $|x - 2| < \delta$

$|1 - 3x - (-5)| < \epsilon$

$|6 - 3x| < \epsilon$

$|-3(x - 2)| < \epsilon$

$3|x - 2| < \epsilon$

$|x - 2| < \frac{\epsilon}{3}$

$\delta = \frac{\epsilon}{3}$ (or smaller)

Of the answer choices,
only (D) $\delta = \frac{\epsilon}{4} \leq \frac{\epsilon}{3}$

Example 4:

Find the limit, then prove it: $\lim_{x \rightarrow -1} \left(\frac{2}{3} - \frac{1}{5}x \right) = \frac{2}{3} - \frac{1}{5}(-1) = \frac{2}{3} + \frac{1}{5} = \frac{13}{15}$

$|f(x) - L| < \epsilon$ for $|x - (-1)| < \delta$

$|x + 1| < \delta$

$\left| \frac{2}{3} - \frac{1}{5}x - \frac{13}{15} \right| < \epsilon$

$15 \cdot \left| -\frac{1}{5}x - \frac{3}{15} \right| < \epsilon \cdot 15$

$|-3x - 3| < 15\epsilon$

$|-3(x + 1)| < 15\epsilon$

$3|x + 1| < 15\epsilon$

$|x + 1| < 5\epsilon$

$\delta = 5\epsilon$

proof:

if $\delta = 5\epsilon$

$\left(\frac{1}{5} \right) |x + 1| < 5\epsilon \left(\frac{1}{5} \right)$

$\left| \frac{1}{5}x + \frac{1}{5} \right| < \epsilon$

$\left| -\frac{1}{5}x - \frac{1}{5} \right| < \epsilon$

$\left| -\frac{1}{5}x - \frac{3}{15} + \frac{2}{3} - \frac{2}{3} \right| < \epsilon$

$\left| \left(\frac{2}{3} - \frac{1}{5}x \right) - \frac{3}{15} - \frac{10}{15} \right| < \epsilon$

$\left| \left(\frac{2}{3} - \frac{1}{5}x \right) - \frac{13}{15} \right| < \epsilon$

$|f(x) - L| < \epsilon \checkmark$