§12.4—Trigonometric Substitution

We have seen integrals similar to $\int x\sqrt{4-x^2}\,dx$. For such an integral, we can integrate quickly by recognizing the pattern ("off" by a -2), or we can do a formal u-substitution, which would replace the old "complex" inside function with a single variable. In this case, we would let $u = 4 - x^2$.

Often, though, integrals such as $\int \sqrt{4-x^2}\,dx$ show up (for instance, when finding the area of a circle or ellipse). This is a much more difficult integral than the first type. For such an integral, we will use a process called **inverse substitution**.

Rather than replacing a complex-looking function with a single variable, we will replace a single variable with a more complex-looking function—making it look more complex in order to make it easier to work with. How will that work? In this case, it will involve **trigonometric substitution**.

The goal of trig sub (for short) is to get rid of the radical in the integrand by way of the Pythagorean Identities, creating a single squared term rather than two terms. Recall that

$$
\boxed{\begin{aligned}
1-\sin^2\theta &= \cos^2\theta \\
1+\tan^2\theta &= \sec^2\theta \\
\sec^2\theta - 1 &= \tan^2\theta
\end{aligned}}
$$

The left side in each of the above identities resembles the form of each radicand in the integrand, with a proper trig sub, we will be able to transform the radicand into something resembling the right side. Let's explore.

Example 1:

Given $\sqrt{4-x^2}$, determine which Pythagorean Identity form from above the radicand resembles, then determine a proper substitution to transform the radicand into a single squared term. Based on your trig substitution, draw a reference triangle and label all three sides in terms of x.

$\sqrt{\text{number}^2 - \text{function}^2}$

resembles $1-\sin^2\theta$

(similar to arcsine integral)

Let $x = 2\sin\theta$

so $\sqrt{4-x^2}$

$= \sqrt{4-(2\sin\theta)^2}$

$= \sqrt{4-4\sin^2\theta}$

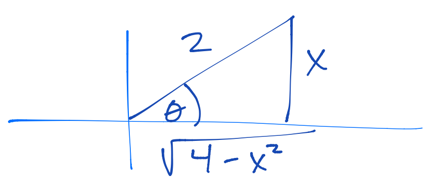
$= \sqrt{4(1-\sin^2\theta)}$

$= \sqrt{4\cos^2\theta}$

$= \boxed{2\cos\theta}$

if $x = 2\sin\theta$

then $\sin\theta = \frac{x}{2}$



So $2\cos\theta$

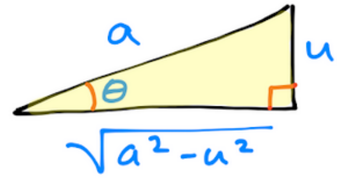
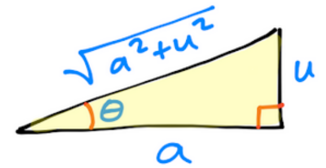
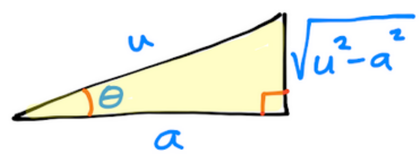
$= 2\left(\frac{\sqrt{4-x^2}}{2}\right)$

$= \sqrt{4-x^2}$ ✓ check!

In general, for the type of radical expression of the form $\sqrt{a^2-u^2}$, where a is any constant and u is any function of x, we can make the substitution $u = a\sin\theta$ to obtain the following

$$\sqrt{a^2-u^2}=\sqrt{a^2-a^2\sin^2\theta}=\sqrt{a^2\left(1-\sin^2\theta\right)}=\sqrt{a^2\cos^2\theta}=a\left|\cos\theta\right|$$

Trig Subs

Expression	Substitution	Identity Used	Triangle Drawn
$\sqrt{a^2-u^2}$	$u=a\sin\theta$	$1-\sin^2\theta=\cos^2\theta$	
$\sqrt{a^2+u^2}$	$u=a\tan\theta$	$1+\tan^2\theta=\sec^2\theta$	
$\sqrt{u^2-a^2}$	$u=a\sec\theta$	$\sec^2\theta-1=\tan^2\theta$	

Note: When we make these substitutions with functions of θ, we are doing so over the principal value ranges of arcsine, arctangent, and arcsecant. This will assure the substitution is a one-to-one function.

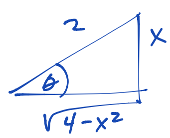
Example 2:

Evaluate $\int\sqrt{4-x^2}\,dx$. You might need some of the Trig Identities at right.

$$\cos^2\theta=\frac{1}{2}\left(1+\cos 2\theta\right)$$
$$\sin^2\theta=\frac{1}{2}\left(1-\cos 2\theta\right)$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta=\cos^2\theta-\sin^2\theta$$

Let $x=2\sin\theta \longrightarrow \sin\theta=\frac{x}{2} \rightarrow \theta=\arcsin\frac{x}{2}$

$dx=2\cos\theta\,d\theta$



$\int\sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta$

$\int 2\cos\theta\cdot 2\cos\theta\,d\theta$

$4\int\cos^2\theta\,d\theta$

$4\int\frac{1}{2}(1+\cos 2\theta)\,d\theta$ * Identity substitutions

$2\int(1+\cos 2\theta)\,d\theta$

$2\left[\theta+\frac{1}{2}\sin 2\theta\right]+C$

$2\theta+\sin 2\theta+C$

*put back in terms of x now

$2\theta+2\sin\theta\cos\theta+C$

$2\arcsin\left(\frac{x}{2}\right)+2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right)+C$

$$\boxed{2\arcsin\left(\frac{x}{2}\right)+\frac{x\sqrt{4-x^2}}{2}+C}$$

Example 3:

Evaluate $\int \frac{\sqrt{16-x^2}}{x^2}dx$

$16-x^2=4^2-x^2$

Let $x=4\sin\theta$

$dx=4\cos\theta\, d\theta$

subbing: $\int \frac{\sqrt{16-16\sin^2\theta}}{(4\sin\theta)^2}\cdot 4\cos\theta\, d\theta$

$\int \frac{\sqrt{16(1-\sin^2\theta)}}{16\sin^2\theta}\cdot 4\cos\theta\, d\theta$

$\int \frac{\sqrt{16\cos^2\theta}}{16\sin^2\theta}\cdot 4\cos\theta\, d\theta$

$\int \frac{\cancel{4}\cos\theta}{\cancel{16}\sin^2\theta}\cdot \cancel{4}\cos\theta\, d\theta$

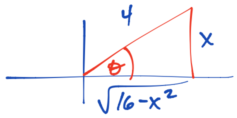
$\int \frac{\cos^2\theta}{\sin^2\theta}d\theta$

$\int \cot^2\theta\, d\theta$ * $1+\cot^2\theta=\csc^2\theta$ so $\cot^2\theta=\csc^2\theta-1$

$\int(\csc^2\theta-1)\,d\theta$

$-\cot\theta-\theta+C$

* if $x=4\sin\theta$ then $\sin\theta=\frac{x}{4}\rightarrow\theta=\arcsin\left(\frac{x}{4}\right)$



$-\cot\theta-\theta+C$

$\boxed{-\frac{\sqrt{16-x^2}}{x}-\arcsin\left(\frac{x}{4}\right)+C}$

Example 4:

Evaluate $\int \frac{dx}{\sqrt{9x^2+1}}$

$9x^2+1$ resembles $\tan^2\theta+1=\sec^2\theta$

$u^2+a^2=(3x)^2+(1)^2$

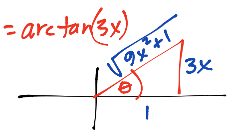
Let $u=3x=1\cdot\tan\theta$

$x=\frac{1}{3}\tan\theta$

$dx=\frac{1}{3}\sec^2\theta\, d\theta$

$\tan\theta=\frac{x}{1/3}=\frac{3x}{1}$

$\theta=\arctan(3x)$



$\int \frac{1}{\sqrt{9\left(\frac{1}{3}\tan\theta\right)^2+1}}\cdot\left(\frac{1}{3}\right)\sec^2\theta\, d\theta$

$\int \frac{1}{\sqrt{\tan^2\theta+1}}\left(\frac{1}{3}\right)\sec^2\theta\, d\theta$

$\int \frac{1}{\sqrt{\sec^2\theta}}\cdot\left(\frac{1}{3}\right)\sec^2\theta\, d\theta$

$\frac{1}{3}\int \frac{\sec^2\theta}{\sec\theta}d\theta$

$\frac{1}{3}\int \sec\theta\, d\theta$

$\frac{1}{3}\ln|\sec\theta+\tan\theta|+C$

subbing back in terms of x: $\boxed{\frac{1}{3}\ln\left|\sqrt{9x^2+1}+3x\right|+C}$

Example 5:

Evaluate $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}dx$

x^2-3 resembles $u^2-a^2 \to$ secant
$u=x,\ a=\sqrt{3}$

Let $x=\sqrt{3}\sec\theta$

$dx=\sqrt{3}\sec\theta\tan\theta$

* change interval of integration

when $x=\sqrt{3}$:
$\sqrt{3}=\sqrt{3}\sec\theta$
$\sec\theta=1$
$\theta=0$

when $x=2$:
$2=\sqrt{3}\sec\theta$
$\sec\theta=\frac{2}{\sqrt{3}}$
$\cos\theta=\frac{\sqrt{3}}{2}$
$\theta=\frac{\pi}{6}$

$$\int_0^{\pi/6}\frac{\sqrt{(\sqrt{3}\sec\theta)^2-3}}{\sqrt{3}\sec\theta}\cdot\sqrt{3}\sec\theta\tan\theta\,d\theta$$

$$\int_0^{\pi/6}\frac{\sqrt{3(\sec^2\theta-1)}}{\cancel{\sqrt{3}}\cancel{\sec\theta}}\cdot\frac{\cancel{\sqrt{3}}\cancel{\sec\theta}\tan\theta}{1}\,d\theta$$

$$\int_0^{\pi/6}\sqrt{3\tan^2\theta}\cdot\tan\theta\,d\theta$$

$$\sqrt{3}\int_0^{\pi/6}\tan^2\theta\,d\theta$$

$$\sqrt{3}\int_0^{\pi/6}\tan^2\theta\,d\theta \quad * \tan^2\theta=\sec^2\theta-1$$

$$\sqrt{3}\int_0^{\pi/6}(\sec^2\theta-1)\,d\theta$$

$$\sqrt{3}(\tan\theta-\theta)\Big|_0^{\pi/6}$$

$$\sqrt{3}\left[\left(\tan\tfrac{\pi}{6}-\tfrac{\pi}{6}\right)-(\tan 0-0)\right]$$

$$\sqrt{3}\left[\frac{\sqrt{3}}{3}-\frac{\pi}{6}\right]$$

$$\boxed{1-\frac{\pi\sqrt{3}}{6}=0.0931003179}$$

There needn't be a radical to use trig sub . . .

Example 6:

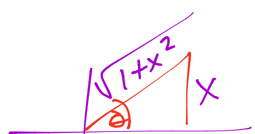
Evaluate $\displaystyle\int \frac{2}{\left(x^2+1\right)^2}dx$

x^2+1 resembles $u^2+a^2 \to$ tangent
$u=x,\ a=1$

Let $x=1\cdot\tan\theta$

$x=\tan\theta \longrightarrow \theta=\arctan x$

$dx=\sec^2\theta\,d\theta$



$\tan\theta=\frac{x}{1}$

$$2\int\frac{1}{((\tan\theta)^2+1)^2}\sec^2\theta\,d\theta$$

$$2\int\frac{\sec^2\theta}{(\tan^2\theta+1)^2}d\theta$$

$$2\int\frac{\sec^2\theta}{(\sec^2\theta)^2}d\theta$$

$$2\int\frac{\cancel{\sec^2\theta}}{\cancel{\sec^2\theta}\cdot\sec^2\theta}d\theta$$

$$2\int\frac{1}{\sec^2\theta}d\theta$$

$$2\int\cos^2\theta\,d\theta \quad * \cos^2\theta=\tfrac{1}{2}(1+\cos2\theta)$$

$$2\int\left(\tfrac{1}{2}(1+\cos2\theta)\right)d\theta$$

$$\int(1+\cos2\theta)\,d\theta$$

$$\theta+\tfrac{1}{2}\sin2\theta+C$$

$$\theta+\tfrac{1}{2}(2\sin\theta\cos\theta)+C$$

$$\theta+\tfrac{1}{2}(2\sin\theta\cos\theta)+C$$

$$\theta+\sin\theta\cos\theta+C$$

Subbing back in: $\arctan x+\left(\dfrac{x}{\sqrt{1+x^2}}\right)\left(\dfrac{1}{\sqrt{1+x^2}}\right)+C$

$$\boxed{\arctan x+\frac{x}{1+x^2}+C}$$