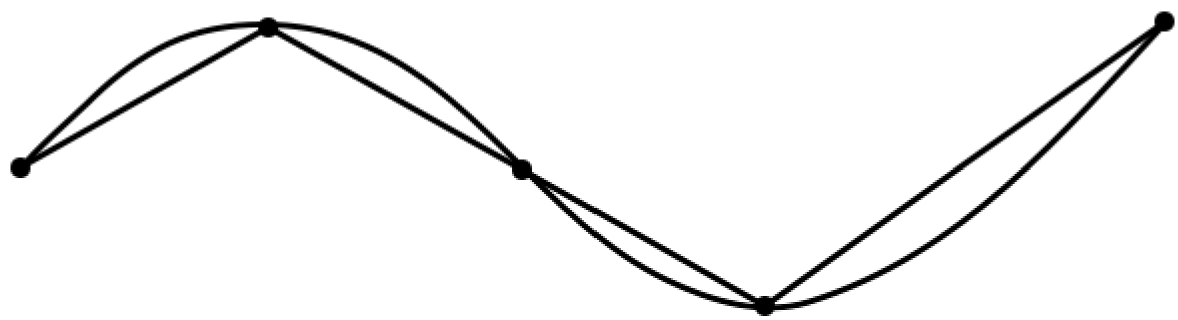
§6.4—Arc Length

If we walk along a curved path with a pedometer or a GPS device, we have a pretty good idea of how far we've gone. If we walk along a curved path and have only the equation of the function along whose path we travel, a **much more likely scenario**, then we can use calculus to find how far we've gone . . . .if we wanted to. Oh, we want to.

We can approximate our distance by dividing our path into several equal partitions and sum the distance between consecutive points. You already know where this is going . . . to achieve better and better approximations, we take smaller and smaller line segments. Voilà! The limit process emerges once again.



This finite process, with the limit attached becomes a very simple integral. Here's how it's derived live!!

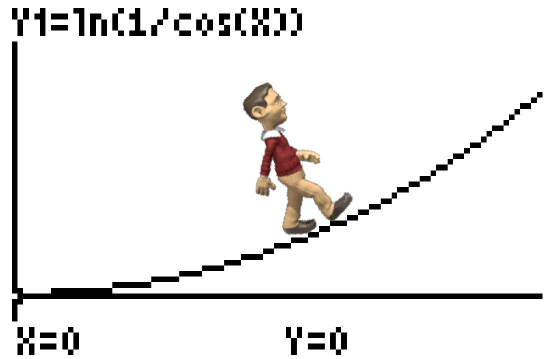
Arc Length, *L* (sometimes *S*)

The length of an arc of a function $y = f(x)$ from $x = a$ to $x = b$ is given by the formula:

$$L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,dx$$

Example 1:

I'm walking along a path defined by $y = \ln(\sec x)$ from $x = 0$ to $x = \frac{\pi}{4}$ miles. At the end of my stroll, how far have I travelled? Give an exact and approximate answer. (And, "Yes!" I only travel along paths that can be mathematically expressed.)



Example 2:

Find the exact length of the curve $y = \frac{4\sqrt{2}}{3}x^{3/2} - 1$ for $0 \le x \le 1$.

This seems easy enough, but sometimes we need to recognize that we are walking along a precarious path filled with craggy cliffs, steep slopes, and jagged rocks.

Because the Arc Length formula relies on the derivative of a function as part of the integrand, any time the derivative of a function fails to exist in our interval of integration (vertical tangent, cusp, discontinuity), we need to be very careful. It also helps to know when to call on a **calculator** or a trained expedition guide.

Fractional exponent alert!

Example 3:
(Calculator Permitted, but discouraged until absolutely necessary) Find the length of the curve $y = \sqrt[3]{x}$ on the interval $[-8, 8]$. What's the problem with this? How can we cleverly circumvent it?

Absolute value alert!

Example 4:
(Calculator Permitted) Find the length of the curve $y = x^2 - 4|x| - x$ from $x = -4$ to $x = 4$.

Sometimes it helps to change all the variables in the very, very beginning. We should be sure to either clearly redefine the variables, or do it "off the record" and report our answer in terms of the original variables.

Example 5:

Determine the length of $x = \frac{2}{3}(y-1)^{3/2}$ between $1 \le y \le 4$.

We also need to be on the look out for not-so-obvious perfect squares.

Example 6:

Find the exact length of the curve $x = \left(y^3/6\right) + 1/\left(2y\right)$ from $y = 1$ to $y = 2$ analytically by antidifferentiation.

Example 7:

A manufacturer of corrugated panel roofing wants to produce "C" panels that are 28 inches wide and 2 inches thick by processing flat sheets of metal as shown in the figure. The profile of the roofing takes the shape the sine wave $y = \sin\left(\frac{\pi}{7}x\right)$. Find the width *w* of a flat metal sheet that is needed to make a 28-inch panel. Assume the process does not stretch the material. Round to three decimals.