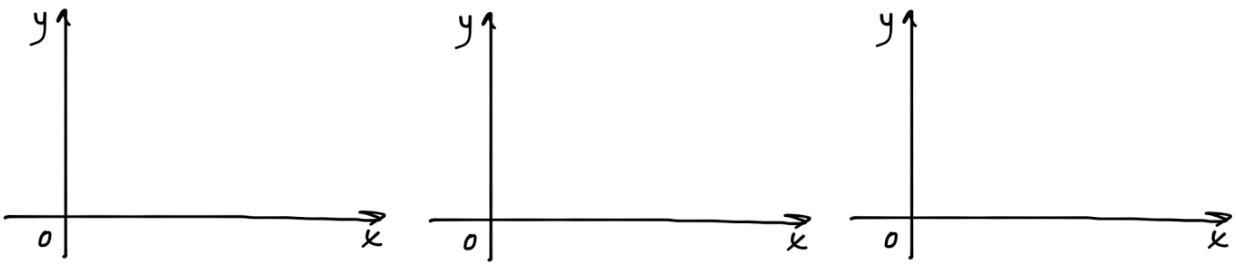
§3.4—Concavity and the Second Derivative Test

When telling the story of what the graph of a function is doing on any interval, the function values give you the y-values on that interval and the first derivative tells you how those y-values are changing. While integral to the tale, these two pieces are only ***part*** of the story. A good mathematical yarn-spinner, for which there is a high demand, will placate the listeners' curiosities by telling them HOW that function is changing (reluctantly/gleefully/nervously/**at an increasing rate/at a decreasing rate/at a constant rate**).

Example 1:

Let $f(x)$ be a function such that $f'(x)>0$ on some interval. Draw three different ways in which the graph of $f(x)$ might appear on this interval. In each case, analyze **how** the **slopes** of the graph of $f(x)$ are changing.



The graphs of the three functions above all increase over the interval, but they **curve** differently, and, therefore, have different **<u>concavity</u>**. One increases at an increasing rate, another at a decreasing rate, and the third increases at a constant rate (functions that increase at a constant rate are linear, boring, and don't require calculus.)

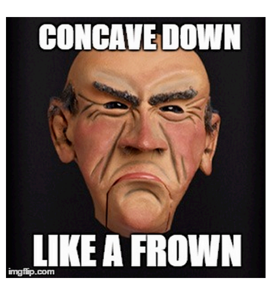
<u>Important Idea</u>: If f' **tells us how the *y*-values of a graph of *f* are changing**, then the derivative of f', or f''**, tells us how the slopes of** f **are changing**.

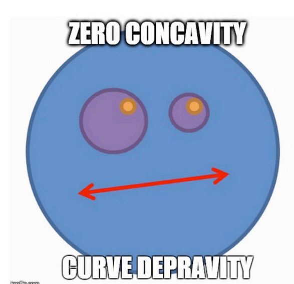
Relation between f'' and concavity on an interval

For a continuous function $f(x)$ on an interval,

- $f''(x)>0 \Leftrightarrow f$ is **<u>concave up</u>** (like a cup) on the interval $\Leftrightarrow$ slopes of f are increasing.
- $f''(x)<0 \Leftrightarrow f$ is **<u>concave down</u>** (like a frown) on the interval $\Leftrightarrow$ slopes of f are decreasing.
- $f''(x)=0 \Leftrightarrow f$ has no curvature/concavity $\Leftrightarrow$ slopes of f are constant.

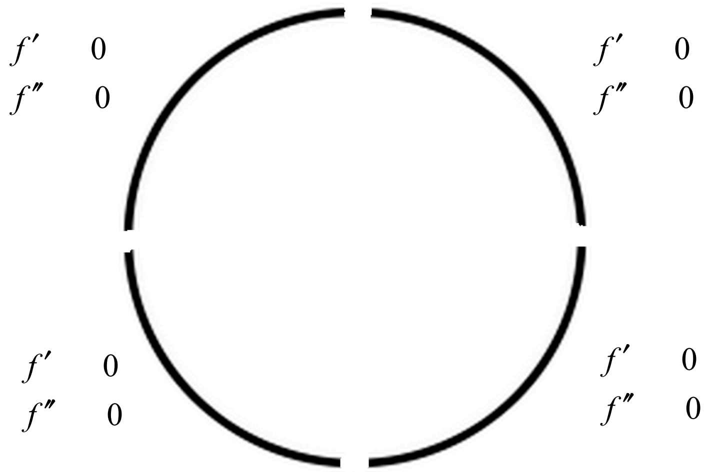
Here's an easy way to remember the curvature of $f(x)$ based on the sign of $f''(x)$.





Putting it together, if we know the signs of both f' and f'' on an interval, we know what the shape of the graph of f looks like.

Example 2:
Determine the signs of f' and f'' for each of the curved segments below. Fill in the inequality.



Knowing the actual function values allows us to place the correct shape in the correct spot on the graph. But before we get into graphing, we first need to discover the **values of x at which a function can change its concavity**.

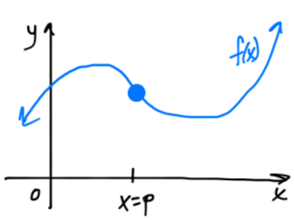
Definition

An x-value, $x = p$, in the domain of a function $f(x)$, is said to be an **inflection value** of $f(x)$ if the graph of $f(x)$ changes from either concave up to concave down at $x = p$ or from concave down to concave up at $x = p$.

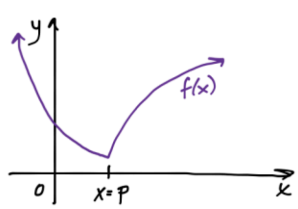
That is, either f'' changes from positive to negative at $x = p$ or f'' changes from negative to positive at $x = p$.

The point $(p, f(p))$ is called the **inflection point**.

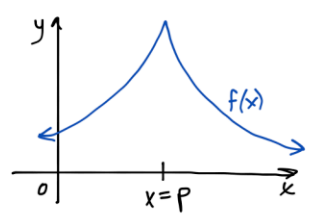
Important note: A graph can also change its concavity at a **discontinuity**, like a vertical asymptote, but if this x-value is not in the domain of the function, it CANNOT and WILLNOT be an inflection value.



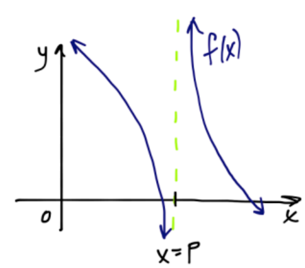
Inflection value at $x = p$



Inflection value at $x = p$



NO inflection value at $x = p$



NO inflection value at $x = p$

Definition

A **possible inflection value**, p.i.v., of a function $f(x)$, is an x-value **in the domain** of $f(x)$ such that either $f''(x)=0$ or $f''(x)=DNE$. These values are essentially critical values of f'.

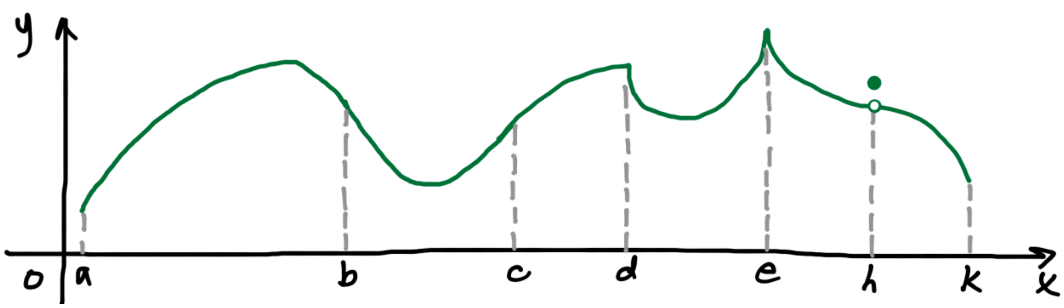
Theorems

- The graph of a function $f(x)$ can change its concavity at either a possible inflection value or a discontinuity.
- An inflection value can only occur at a possible inflection value, p.i.v., but not every p.i.v. is an inflection value.

Test for intervals of concavity

1. Find any discontinuities of a function $f(x)$.
2. Identify any p.i.v.s in the domain of $f(x)$ by finding where $f''(x)=0$ or $f''(x)=DNE$
3. Using a number line chart, test the intervals by finding the sign of f'' on all intervals between any p.i.v.'s and discontinuities.
4. On the intervals where $f''(x)>0$, $f(x)$ is **concave up** (like a cup) on that interval. AND
   On the intervals where $f''(x)<0$, $f(x)$ is **concave down** (like a frown) on that interval.

Example 3:
The graph of a function $f(x)$ defined everywhere on the interval $[a,k]$ is given below.



(a) Give the possible inflection values, p.i.v.s.

(b) At each p.i.v., determine if $f(x)$ has an inflection value. Justify.

(c) List the open intervals on which the graph $f(x)$ is concave up, concave down, and/or constant.

Just like the **first derivative number line chart** can be used to determine intervals of increasing/decreasing as well as local extrema, so too can the **second derivative number line chart** be used for determining intervals of concavity as well as actual inflection points!!

Important ideas to reiterate:

* Not every p.i.v. is an inflection value.
* Concavity can change at a discontinuity, such as a VA, but it won't be an inflection point.
* To find **possible inflection values** (p.i.v.'s), find any $c \in D_f$ such that $f''(c) = 0$ or $f''(c)$ is undefined at $x = c$ (as long as $f(c)$ is defined).
* A chart can help you efficiently test for concavity by determining the sign of f'' in between p.i.v.'s and **discontinuities**.

Example 4:
For each of the following functions, find (i) the p.i.v.s, (ii) the inflection values with justification, and (iii) the open intervals of concavity.

(a) $y = 3 + \sin x \quad x \in [0, 2\pi]$

(b) $f(x) = 6(x^2 + 3)^{-1}$

(c) $g(x) = \dfrac{x^2 + 1}{x^2 - 4}$

(d) $f(x) = x^4 - 4x^3$

(e) $y = e^{-x^2}$

(f) $g(x) = x^4$

(g) $f(x)=\sqrt[3]{x}$

(h) $y=\begin{cases}\sqrt{-x}, & x\le 0\\ x^2, & x>0\end{cases}$

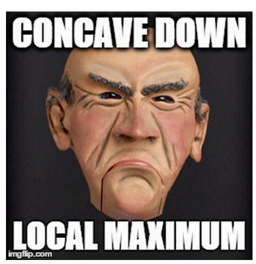
We will now return briefly to something from the previous section: Relative/Local Extrema. It turns out, there is a nice little application of the second derivative that may be helpful in determining these extreme values.

The Second Derivative Test (for Relative Extrema)

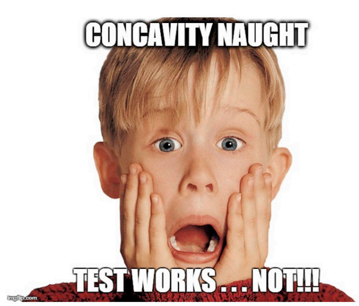
Let f be a function such that $x=c$ is a **critical value** of f such that $f'(c)=0$. If $f''(x)$ exists on an open interval containing $x=c$, then

1. If $f''(c)>0$, then $(c, f(c))$ is a relative minimum.

2. If $f''(c)<0$, then $(c, f(c))$ is a relative maximum.



3. If $f''(c)=0$, then the test fails, and the First Derivative Test must be used.



Example 5:

Find the relative extrema for $f(x) = -3x^5 + 5x^3$ using

(a) The First Derivative Test. Justify.

(b) The Second Derivative Test (if possible). Justify.

Example 6:

Selected values for a twice-differentiable function $f(x)$, continuous on , $-3 \le x \le 5$ is given below along with selected value for $f'(x)$, and $f''(x)$.

x	$f(x)$	$f'(x)$	$f''(x)$
-3	0	5	1
2	2	0	4
5	7	-1	0

(a) Is it safe to say that $f(x)$ is concave up on the interval $-3 < x < 5$? Why or why not?

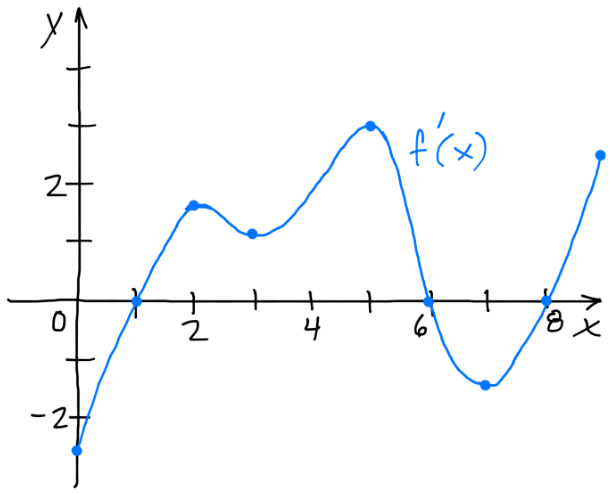
(b) (Review) Explain why there must be a $z \in (-3,5)$ such that $f(z) = 6$.

(c) (Review) Explain why there must there must be a $w \in (-3,5)$ such that $f'(w) = \frac{7}{8}$.

(d) Does $f(x)$ have a local maximum, local minimum, or neither at $x = 2$? Justify.

Sometimes we need to be able to answer questions regarding a function $f(x)$ and its concavity from a graph of $f'(x)$.

Example 7:
The graph of the derivative f' of a **continuous function** $f(x)$ on $[0,9]$ is shown below. Answer the following questions in complete sentences.



(a) On what open interval(s) is f decreasing? Justify.

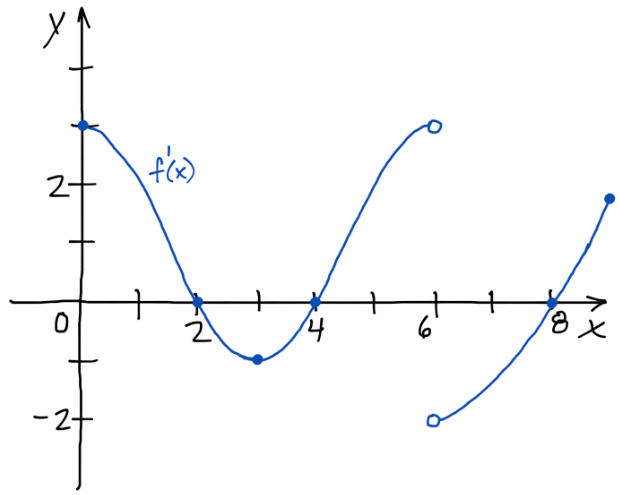
(b) At what value(s) of x does f have a local maximum or minimum? Justify.

(c) On what intervals is f concave up? Justify.

(d) State the inflection values of f. Justify.

(e) Assuming that $f(0)=0$, sketch a graph of f. If possible, determine the x-value(s) at which f attains its maximum and/or minimum value(s).

Example 8:
The graph of the derivative f' of **a continuous function** $f(x)$ on $[0,9]$ is shown below. Answer the following questions in complete sentences.



(a) On what open interval(s) is f increasing? Justify.

(b) At what value(s) of x does f have a local maximum or minimum? Justify.

(c) On what intervals is f concave down? Justify.

(d) State the inflection values of f. Justify.

(e) Assuming that $f(0)=0$, sketch a possible graph of f. Determine the x-value(s) at which f attains its maximum and/or minimum value(s).