§3.1—Extrema on an Interval

Now that we know how to find derivatives, we will apply them to analyzing the graphs of functions.

Definition: (Absolute/Global) Extrema/Extreme *y*-values

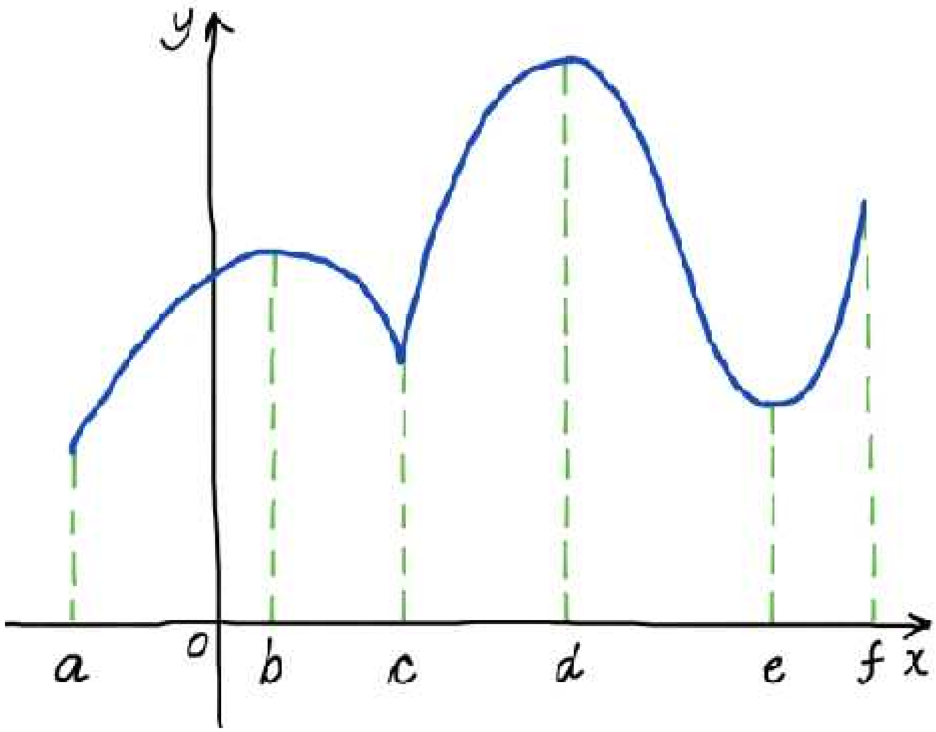
If f is a function on an interval *I*, then $y = f(c)$ is the

I. (Absolute/Global) **Maximum** on *I*, IFF $f(c) \geq f(x)$ for all *x* in *I*.

II. (Absolute/Global) **Minimum** on *I*, IFF $f(c) \leq f(x)$ for all *x* in *I*.

Notice we cannot say that the maximum is the *y*-value that is the BIGGEST on an interval, but rather we must say it's the *y*-value for which there are none bigger. Similarly with the minimum.

Example 1:
The graph of $g(x)$ is given below. Determine the extrema of $g(x)$ on the interval $x \in [a, f]$.



Example 2:

Sketch the following functions on the given interval, then, determine the extrema, if they exist.

(a) $f(x) = x^2 + 1$ on $[-1, 2]$

(b) $f(x) = x^2 + 1$ on $(-1, 2)$

(c) $f(x) = \begin{cases} x^2 + 1, & x \neq 0 \\ 2, & x = 0 \end{cases}$ on $[-1, 2]$

(d) $f(x) = \begin{cases} x^2 + 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$ on $[-1, 2]$

(e) $f(x) = \sin x$ on $[0, 4$®$]$

(f) $f(x) = 2 - 3x$ on $[-1, 2]$

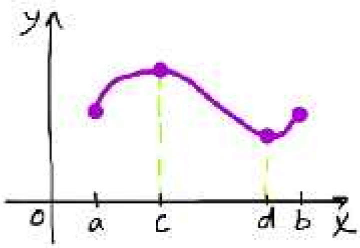
(g) $f(x) = 3$ on $[-2, 4]$

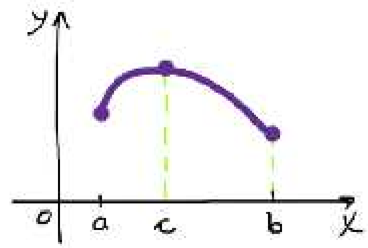
Under what conditions do you think a function will have BOTH and absolute max AND min?

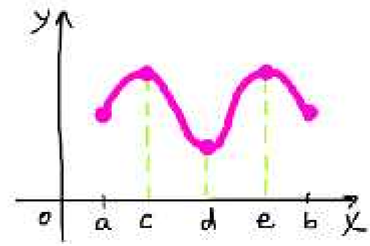
The Extreme Value Theorem (EVT)

If f is continuous on a closed interval $[a,b]$, then f has both a maximum value and a minimum value on the interval $[a,b]$.

Here are some examples of functions on $[a,b]$ where the EVT applies.





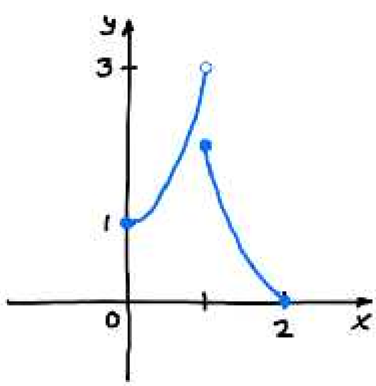


If the hypothesis ("if" part) is not met, either the continuity or the closed interval part, there is no guarantee of the conclusion, but a max, min, or both still may exist, they are both just not guaranteed.

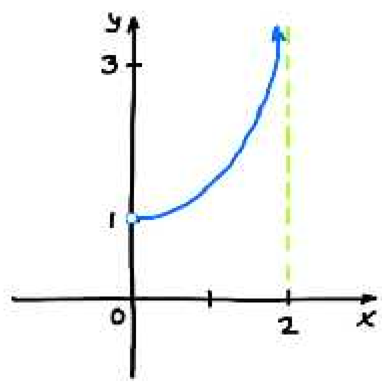
Example 3:

Determine if the EVT applies for each of the function on the interval $[0,2]$. If so, find the extrema. If not, explicitly state why, then determine if the function happens to still have extrema on the interval.

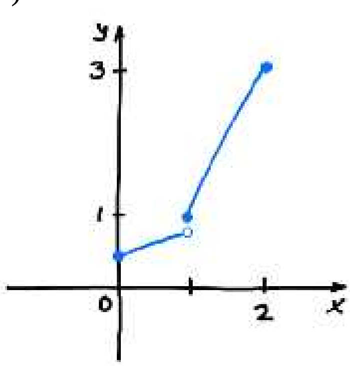
(a)



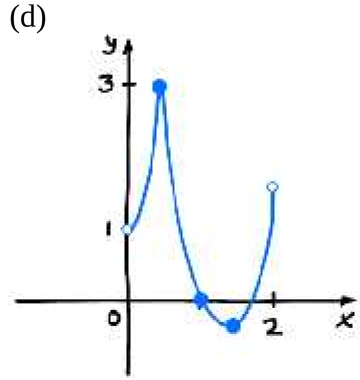
(b)



(c)



(d)



So now that we know WHEN we will have both a max and a min, how do we find them from the equation of a function and not the graph? The theorem doesn't tell us ***that***! The next question then is: **at what different values of x can extreme values occur?**

Definition: Critical Value

A **critical value** of a function f is a value $x = c$ in the domain of f such that either

$$f'(c) = 0 \text{ or } f'(c) = DNE$$

If $x = c$ is a critical value, then $(c, f(c))$ is a critical point.

Theorem

(Absolute/Global) Extrema can only occur on at a **critical value** OR at an **endpoint** of an interval.

Before we set out to find extrema analytically by analyzing the equation of a function, we need to address another type of extreme value.

Definition: Relative/Local Extrema

If there exists an **open** interval containing $x = c$, then

I. If immediately to the left of $x = c$ and immediately to the right of $x = c$ there are no y-values **greater** than $f(c)$, then $f(c)$ is a **relative/local maximum** of f.

II. If immediately to the left of $x = c$ and immediately to the right of $x = c$ there are no y-values **smaller** than $f(c)$, then $f(c)$ is a **relative/local minimum** of f.

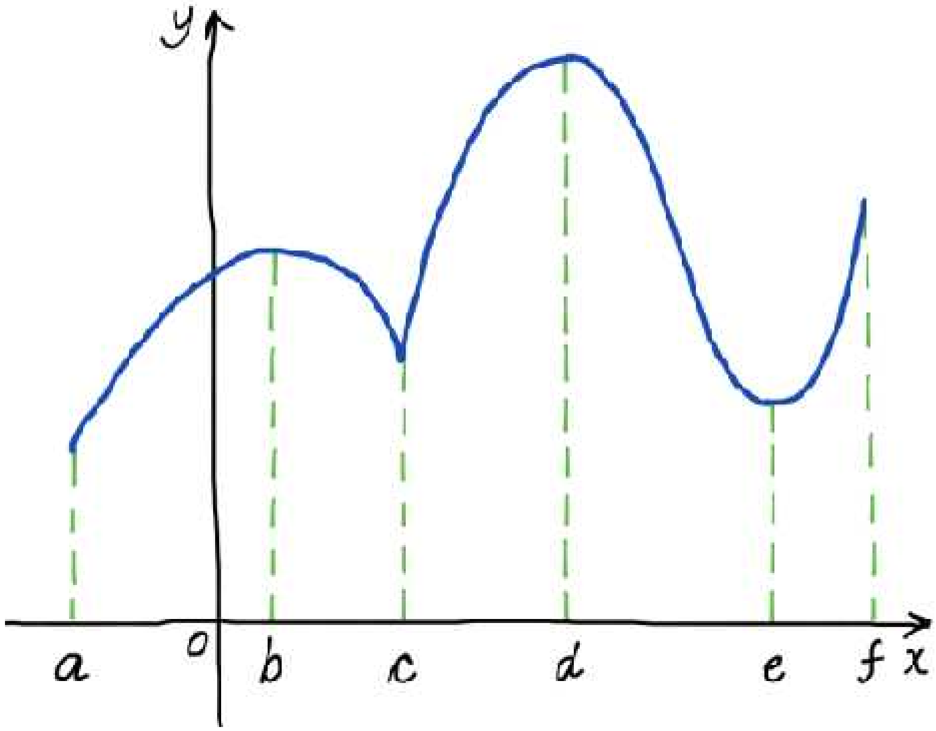
Theorem

Relative/Local Extrema can only occur on at a **critical value** on an **OPEN** interval.

NOTE I: This theorem does NOT say that there is a relative/local extreme value at every critical value (for example, $f(x) = x^3$ at $x = 0$), but rather, every critical value is the potential, prospective, possible location of a relative/local extreme value.

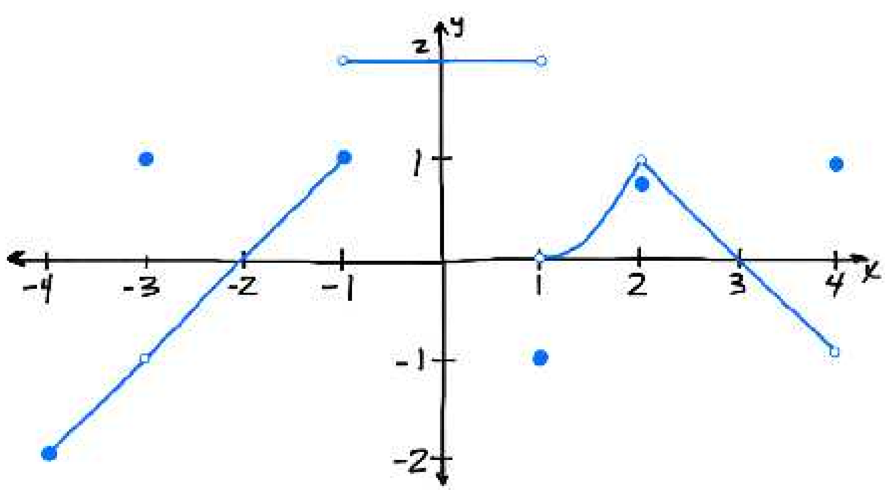
***** NOTE II: Relative/Local extrema CANNOT occur at an endpoint of an interval. They cannot live at the end of a cul-de-sac, but rather along a through street.***

Example 4:
Identify all the critical values of the graph of $g(x)$ shown below, then determine whether a local max, local min, or neither occur at that critical value.



Relative Extrema may not always be at nice, tidy hills and valleys, and can even occur at points of discontinuity.

Example 5:
The graph of $h(x)$ is given below. Find all the critical values of $h(x)$ on the interval $[-4, 4]$, then determine if a local max, local min, or neither occur at each.



Example 6:
Find the **domain** and the **critical values** of each of the following functions:

(a) $f(x)=\frac{1}{2}x^4-x^3-x^2+2$ (b) $g(x)=2-|x-4|$ (c) $f(x)=x^{3/5}(4-x)$.

We are now ready to analytically determine the **absolute extrema** guaranteed by the EVT given the equation of a function.

Closed Interval Method for finding Extrema

To find absolute extrema of a continuous function $f(x)$ on a closed interval $[a,b]$.

1. Identify the endpoints.
2. Identify any critical values in (a,b)—**verify they are in the domain & the specified interval!!**
3. Find the function values, $f(x)$, at both endpoints and at all critical values in (a,b).
4. The largest value will be the max. The smallest value will be the min.
5. Answer the question asked in a complete sentence.

It's like being a contestant on the hit game show "Extrema!!" To be a contestant on the show, you must be an endpoint of an interval or a critical value on the interval.

Who will be the biggest y-value? Who will be crowned the Global Maximum??? Will it be contestant A, the left endpoint? Will it be contestant B, the right endpoint? Or will it be contestant C, the critical value?

In the picture at right, the redhead is the left endpoint of an interval of a continuous function, the bald guy is the right endpoint of an interval, and the fist-pumping gal is a critical value on the interval, and the contestants' scores represent the function value at their location. Who will win the title of

- Absolute Maximum?
- Absolute Minimum?
- Who gets the parting gift of a broken toaster for participating?

Example 7:

Determine if the EVT applies. If so, find the extrema of $f\left(x\right)=3x^4-12x^3$ on the interval $\left[-1,2\right]$.

Example 8:

Determine if the EVT applies. If so, find the extrema of $f\left(x\right)=2x-3x^{2/3}$ on the interval $\left[-8,1\right]$

Example 9:

Determine if the EVT applies. If so, find the extrema of $f\left(x\right)=2\sin x-\cos 2x$ on $\left[0,\frac{3®}{2}\right]$.

What do we do if we want to find extrema, the EVT does not apply, no interval is given? Use all the tools at your disposal, including sketching a graph.

Example 10:
Find the extrema of each of the following functions over their domain by sketching their graphs:

(a) $f(x) = \dfrac{1}{\sqrt{4-x^2}}$

(b) $f(x) = \begin{cases} 5-2x^2, & x \le 1 \\ x+2, & x > 1 \end{cases}$

Sometimes we need to use our calculator's number-crunching ability to **solve equations by finding *x*-intercepts for us.**

Example 11:
Using your calculator's equation solving capability (not just its max/min finding ablity), find the extrema of the $f(x) = xe^{-x} - 2x$ on the interval $[-1,1]$. Be sure to show the equation you're solving and you justification via the Closed Interval Argument.