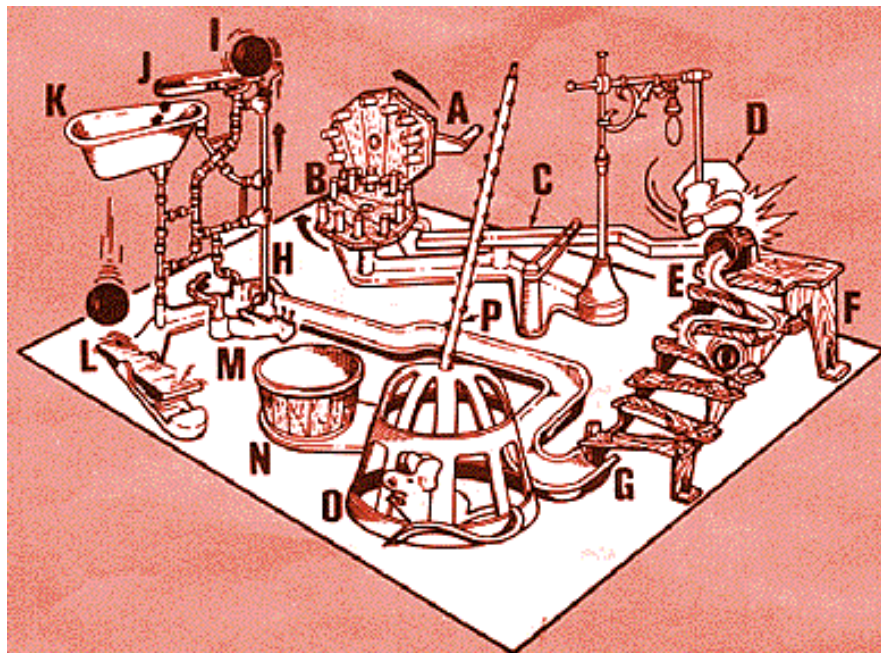


§2.6—The Chain Rule

If you thought the power rule was powerful, it has nothing on the Chain Rule.



Remember the board game “Mousetrap?” The Chain Rule is actually so named because it is similar to a **chain reaction**, whereby one action triggers another, which triggers another, which triggers another, etc.



In the diagram above, the turning of the crank at point *A* travels all the way through the board, eventually trapping of the mouse at point *P*. We want to know what the change in *P* is with respect to *A*.

Using Leibniz notation, we can represent this chain reaction as such

$$\frac{dP}{dA} = \frac{dB}{dA} \cdot \frac{dC}{dB} \cdot \frac{dD}{dC} \cdot \frac{dE}{dD} \cdot \frac{dF}{dE} \cdot \frac{dG}{dF} \cdot \frac{dH}{dG} \cdot \frac{dI}{dH} \cdot \frac{dJ}{dI} \cdot \frac{dK}{dJ} \cdot \frac{dL}{dK} \cdot \frac{dM}{dL} \cdot \frac{dN}{dM} \cdot \frac{dO}{dN} \cdot \frac{dP}{dO}$$

The Chain rule is used to differentiate composite functions, that is, functions (other than the trivial *x*) within other functions. Here’s what type of functions the chain rule will allow us to differentiate.

Without the Chain Rule	With the Chain Rule
$y = x^2 - 1$	$y = \sqrt{x^2 - 1}$
$y = \sin x$	$y = \sin 5x$
$y = 3x - 2$	$y = (3x - 2)^7$
$y = x - \tan x$	$y = x - \tan x^2$

The Chain Rule

If $y = f(u)$ is a differentiable function of u , and $u = g(x)$ is a differentiable function of x , then

$y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

It's like peeling an onion, one layer at a time from the outside in, except we're integrating one function at a time, from the outside in, multiplying by successive derivatives of inside functions. The multiplication symbol, \cdot , can be thought of as a place where to "links" in the chains meet. In the end, there should be as many chain "links" as there are embedded functions.

It may be helpful to think of the chain rule as "unpacking boxes." In doing so, you must "unpack" all the boxes as you get to them.

When using the Chain Rule, it is vitally important to rewrite if necessary so you can clearly identify the layers of the function.

Example 1:

Evaluate the following:

(a) $\frac{d}{dx}[\sqrt{x^2 - 1}]$

(b) $\frac{d}{dx}[\sin 5x]$

(c) $\frac{d}{dx}[(3x - 2)^7]$

(d) $\frac{d}{dx}[x - \tan x^2]$

(e) $\frac{d}{dx}\left[\frac{1}{x+1}\right]$

(f) $\frac{d}{dx}[\sin^2 3x]$

(g) $\frac{d}{dx}[x \sec(x^2 + x + 1)]$

(h) $\frac{d}{dm}[\sec(4 - \cos^3 5m)]$

(i) $\frac{d}{dx}\left[\left(\frac{3x-1}{x^2+3}\right)^2\right]$

Example 2:

Find all x -values on the graph of $f(x) = \sqrt[3]{(x^2 - 1)^2}$ for which $f'(x) = 0$ and/or for which $f'(x)$ does not exist.

Example 3:

If $f(x) = \frac{1}{(1-2x)^3}$,

(a) Find the domain of $f(x)$.

(b) Show that the slope of every line tangent to the curve of $f(x)$ is positive. What conclusion about the behavior of the graph of $f(x)$ can you draw from this?

Example 4:

In each of the following, find $\frac{dy}{dx}$, then simplify $\frac{dy}{dx}$ by **factoring out the least powers**.

(a) $y = x^2 \sqrt{1-x^2}$

(b) $f(x) = \frac{x}{\sqrt[3]{x^2+4}}$

Example 5:

For each function, rewrite if necessary, find $f'(x)$, then simplify.

(a) $f(x) = \cos 5x^2$

(b) $f(x) = (\cos 5)x^2$

(c) $f(x) = \cos(5x)^2$

(d) $f(x) = \cos^2 5x$

(e) $f(x) = \frac{1}{\sqrt{\cos 5x}}$

(f) $f(x) = \cos 5$

Example 6:

Find the equation of the tangent line to the graph of $f(x) = 2 \sin x + \cos 2x$ at $x = \pi$.

Example 7:

Suppose that the differentiable functions f and g and their derivatives have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	$1/3$	-3
3	3	-4	2π	5

Evaluate the derivatives with respect to x of the following combinations at the given value of x .

(a) $2f\left(\frac{3x}{2}\right)$ at $x = 2$

(b) $f(x) + g(x)$ at $x = 3$

(c) $f(x) \cdot g(x)$ at $x = 3$

(d) $\frac{f(x)}{g(x)}$ at $x = 2$

(e) $f(g(x))$ at $x = 2$

(f) $\sqrt{f(x)}$ at $x = 2$

(g) $\frac{1}{g^2(x)}$ at $x = 3$

(h) $\sqrt{f^2(x) + g^2(x)}$ at $x = 2$

Example 8:

Use the fact that $f(x) = |x| = \sqrt{x^2}$ and the chain rule to find $\frac{d}{dx}[|x|]$. For what values of x is $f(x)$ differentiable? What is the domain of $f'(x)$?