

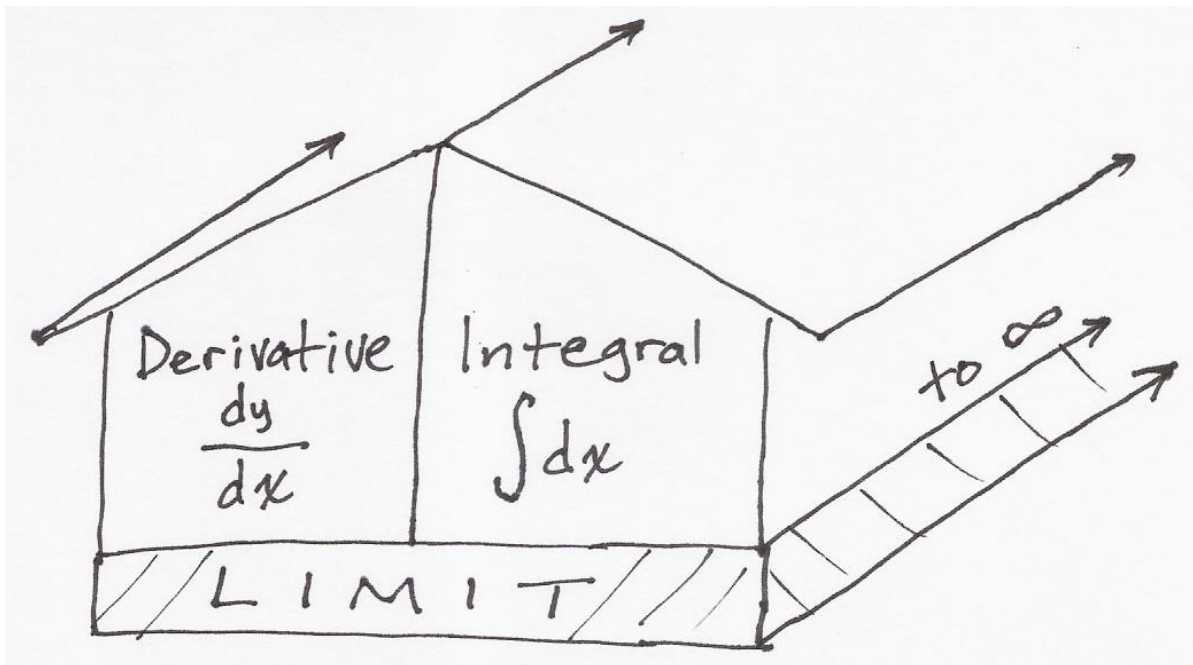
# §1.1—Limits & Continuity

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What do you see below?



We are building the “House of Calculus,” one side at a time . . . and we need a solid FOUNDATION.



**Example 1:**(Calculator) For  $f(x) = x^2$ 

(a) fill in the following chart

$x$	2.9	2.99	2.999	3.1	3.01	3.001
$f(x)$						

(b) What do these values tell us about  $f$  in the **neighborhood** of  $x = 3$ ?(c) Based on the chart above, what do you think is the value of  $f(3)$ ?(d) What IS  $f(3)$ ?**Example 2:**(Calculator) If  $g(x) = \frac{x^3 - 3x^2}{x - 3}$ ,

(a) fill in the following chart

$x$	2.9	2.99	2.999	3.1	3.01	3.001
$g(x)$						

(b) What do these values tell us about  $g$  in the **neighborhood** of  $x = 3$ ?(c) Based on the chart above, what do you think is the value of  $g(3)$ ?(d) What IS  $g(3)$ ?

(e) Is there a still a way to mathematically communicate the result from the chart above?

This interruption to the flow of the graph of  $g$  in Example 2 is called a **removable point discontinuity**, or a **hole** in the graph of  $g$ .

Simply evaluating a function at a particular value is insufficient for understanding the behavior of some types of functions at that point, especially functions with discontinuities at those points. There is, therefore, a need to come up with another method that will circumvent the possibility of going directly to a location, but rather **approaching** that location from either side of it. This is the limit, and it has its own notation as you will see

### The **Limit** is a Notion of Motion

#### **Example 3:**

Suppose you wanted to safely and smooooothly cross a chasm in your car. What three things would you need? Would you be able to do so in the situation depicted in the photo to the right? What is missing?



(a) From the equation of the function, how could you algebraically determine that a function has a **hole (no bridge)** at a particular  $x$ -value?

(b) If you were to build the bridge (**fill in the hole**) at that  $x$ -value, how could you then determine the exact vertical location ( **$y$ -value**) to build it to make crossing the chasm in a car a continuously smooth undertaking?

As you can see, at a particular  $x$ -value, the existence or non-existence of the **bridge (y-value/function value)** is independent of the existence of the **roads leading in from either side (the limit values from the left and right)**. This is exactly why the limit is so important—it gives us a way to talk about the activity in the neighborhood of a point whether or not the **function value/y-value/graph (bridge)** exists at that point or not.

**Example 4:**

For  $g(x) = \frac{x^3 - 3x^2}{x - 3}$ , algebraically determine the coordinate of  $(x, g(x))$  **the removable point discontinuity**, then use **limit notation** to describe what is happening as the discontinuity is approached from either side.

### DEFINITION: General Limit

$$\lim_{x \rightarrow c^-} f(x) = L = \lim_{x \rightarrow c^+} f(x) \Leftrightarrow \lim_{x \rightarrow c} f(x) = L$$

The theorem above essentially says that the limit will exist *if and only if* the two roads coming in from either side are aligning with each other, irrespective of whether there is a bridge connecting the two roads.

The **limit** describes the **y-values** to which the **roads** are leading! The **function value** pertains to the **y-value** of the **bridge** itself at a single point.

**Example 5:**

For  $f(x) = \frac{x^2 + 4x + 3}{x^2 - 3x - 4}$ , algebraically determine the following:

(a)  $f(-1)$

(b)  $\lim_{x \rightarrow -1} f(x)$

(c)  $\lim_{x \rightarrow -1^-} f(x)$

(d)  $\lim_{x \rightarrow -1^+} f(x)$

(e) Based on your answers above, what type of discontinuity does the function have at  $x = -1$ ? Justify. In a full sentence, name the type and **coordinate**  $(x, y)$  of this discontinuity.

If our goal is to safely and **smoothly** drive across the chasm in our car, what relationship among the **road leading in**, the **bridge**, and the **road leading out** must there be?



With the limit notation, we now have a way to define **continuity at a point**.

### DEFINITION: Continuity at a point (HUGELY IMPORTANT)

A function  $f(x)$  is **continuous at a point**  $x = c$  if

$$\lim_{x \rightarrow c^-} f(x) = f(c) = \lim_{x \rightarrow c^+} f(x)$$

that is

$$\text{road in} = \text{bridge} = \text{road out}$$

NOTES ON JUSTIFICATION:

1. To justify, use mathematical evidence with correct notation to make a claim in a complete sentence.
2. A function is continuous at a point ( $x$ -value) if and only if ALL THREE of the above EXIST and are the SAME value ( $y$ -value).
3. If any one of the three above DO NOT EXIST or if any TWO of the above exist but are NOT EQUAL to each other, then the function is NOT continuous at that  $x$ -value.
4. Do not get confused with the LIMIT DEFINITION (ROADS ONLY) and the 3-STEP DEFINITION OF CONTINUITY!!! **A ROAD IS NOT A BRIDGE!!**

We can now formally define our discontinuity in terms of the limit!

### DEFINITION: Removable Point Discontinuity

If for some function  $f(x)$  at  $x = c$ ,

$$\lim_{x \rightarrow c} f(x) = L \text{ but either } f(c) \text{ is undefined or } f(c) \neq L$$

$\Leftrightarrow$

$f(x)$  has a removable point discontinuity (hole) at  $(c, L)$

**Example 6:**

Using the 3-step definition of continuity, determine whether or not the graph the following function,  $h(x)$ , is continuous at the following  $x$ -values. Justify.

If the function is not continuous at the indicated value, name the type and **coordinate**  $(x, y)$  of this discontinuity.

$$h(x) = \frac{x^3 - 4x}{x^2 + 8x - 20}$$

(a)  $x = 2$

(b)  $x = 0$

(c)  $x = -10$

We know that the **function y-value** and the **limit y-value** exist independently of one another, but the one-sided limits may also exist independently of each other, that is, **the two roads do not necessarily need to line up with each other!!!**

**Example 7:**

For  $f(x) = \begin{cases} x^2, & x < 2 \\ 2x-1, & x > 2 \end{cases}$ , **algebraically/analytically** determine the following:

(a) The domain of  $f(x)$

(b)  $\lim_{x \rightarrow 2^-} f(x)$

(c)  $\lim_{x \rightarrow 2^+} f(x)$

(d)  $\lim_{x \rightarrow 2} f(x)$

(e)  $f(2)$

(f) continuity of  $f$  at  $x = 2$ . Justify.

(g) the type of discontinuity at  $x = 2$

(h) the graph of the function  $f(x)$ .

## DEFINITION: Non-Removable Jump Discontinuity

If for some function  $f(x)$  at  $x = c$ ,

$$\lim_{x \rightarrow c^-} f(x) = L_1 \text{ and } \lim_{x \rightarrow c^+} f(x) = L_2 \text{ where } L_1 \neq L_2 \text{ (regardless of } f(c))$$

$\Leftrightarrow$

$f(x)$  has a non-removable jump discontinuity at  $x = c$



### Example 8:

Using the definition of continuity at a point, discuss the continuity of the following function **over its entire domain**. Justify.

$$f(x) = \begin{cases} x+2, & x < -2 \\ \frac{x^2}{2}, & -2 \leq x \leq 2 \\ 2, & x > 2 \end{cases}$$

The 3-step definition of continuity at a point can also provide us with a system of equations needed to find unknowns.

**Example 9:**

If  $f(x) = \begin{cases} ax^2 - b, & x < -1 \\ 4, & x = -1 \\ 2ax + b, & x > -1 \end{cases}$ , find the values of  $a$  and  $b$  such that  $f(x)$  is continuous at  $x = -1$ .

Before we move on to explore other discontinuities, it's worth noting something about particular functions that we KNOW to be continuous (such as polynomials, sine, cosine, exponential, etc.)

**FACT: General Limit Definition (backwards)**

**If a function is continuous at a point, then the function  $y$ -value and the limit  $y$ -value are the same at that  $x$ -value!**

$$\text{That is, } \lim_{x \rightarrow c} f(x) = f(c).$$

**Example 10:**

Evaluate each of the following limits:

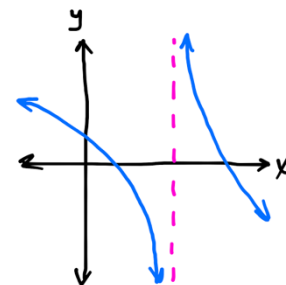
(a)  $\lim_{x \rightarrow 1} (4x^5 - 7x^4 + 3x^3 - x^2 + x - 5)$

(b)  $\lim_{x \rightarrow \frac{5\pi}{3}} \cos x$

(c)  $\lim_{x \rightarrow e} \frac{\ln x}{2}$

In **Example 6**, we saw another type of discontinuity—a vertical asymptote (VA), officially called a **Non-Removable Infinite Discontinuity**.

Similar to a jump discontinuity, the limit will always fail to exist at a VA, but for a very different reason. As we approach a vertical asymptote from either side, there are only two options—**go down forever to negative infinity or go up forever to infinity ... neither of which is a limit!**



## DEFINITION: Non-Removable Infinite Discontinuity

For some function  $f(x)$  at  $x = c$ ,

$$\lim_{x \rightarrow c^+} f(x) = \infty \text{ or } \lim_{x \rightarrow c^+} f(x) = -\infty \text{ OR } \lim_{x \rightarrow c^-} f(x) = \infty \text{ or } \lim_{x \rightarrow c^-} f(x) = -\infty$$

$\Leftrightarrow$

$f(x)$  has a non-removable infinite discontinuity (VA) at  $x = c$ .

### Example 11:

Evaluate for each of the following functions: P.S. Don't use a calculator if you don't have to. P.S.S. You don't have to.

I.  $h(x) = \ln x$     (a)  $h(0) =$     (b)  $\lim_{x \rightarrow 0^-} h(x) =$     (c)  $\lim_{x \rightarrow 0^+} h(x) =$     (d)  $\lim_{x \rightarrow 0} h(x) =$

II.  $f(x) = \frac{1}{x-3}$     (a)  $f(3) =$     (b)  $\lim_{x \rightarrow 3^-} f(x) =$     (c)  $\lim_{x \rightarrow 3^+} f(x) =$     (d)  $\lim_{x \rightarrow 3} f(x) =$

III.  $g(x) = \frac{1}{(x+2)^2}$     (a)  $g(-2) =$     (b)  $\lim_{x \rightarrow -2^-} g(x) =$     (c)  $\lim_{x \rightarrow -2^+} g(x) =$     (d)  $\lim_{x \rightarrow -2} g(x) =$

IV.  $K(x) = \sec x$     (a)  $K\left(\frac{\pi}{2}\right) =$     (b)  $\lim_{x \rightarrow \frac{\pi}{2}^-} K(x) =$     (c)  $\lim_{x \rightarrow \frac{\pi}{2}^+} K(x) =$     (d)  $\lim_{x \rightarrow \frac{\pi}{2}} K(x) =$

**Example 12:**

If  $f(x) = \begin{cases} x^2 + x - 6, & x \neq 2 \\ k, & x = 2 \end{cases}$  is continuous for all  $x \neq \frac{1}{2}$

(a)  $\lim_{x \rightarrow -\frac{1}{2}^+} f(x) =$

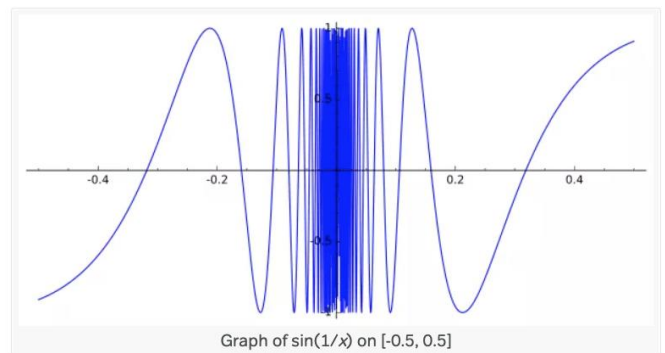
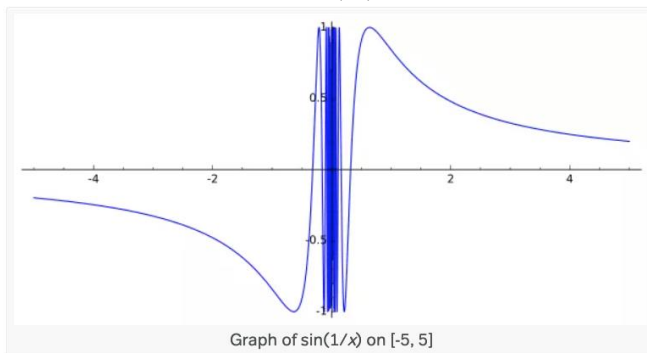
(b)  $\lim_{x \rightarrow 1} f(x) =$

(c) What is the value of  $k$ ?

There is one last type of discontinuity that can be a little tricky. It's called an **oscillating discontinuity**.

**Example 13:**

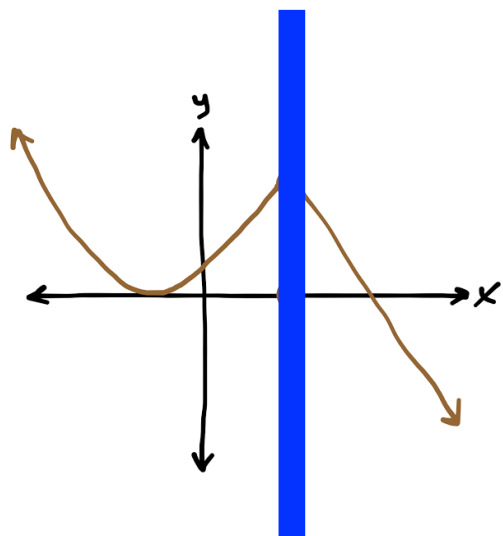
The graph of  $f(x) = \sin\left(\frac{1}{x}\right)$  is given below on two different horizontal scales.



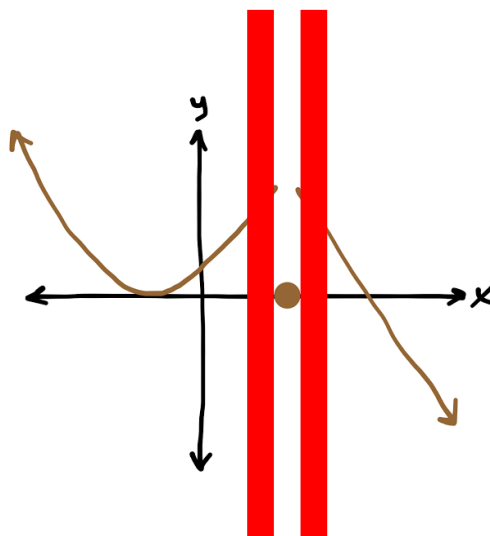
(a) What do you notice about the graph in the neighborhood of  $x = 0$ ?

(b) What is  $\lim_{x \rightarrow 0} f(x)$ ?

Hopefully you're getting comfortable with this new idea of a limit value and how it is categorically different from a function value. Additionally, you've no doubt become aware of how important both the limit value and the function value to the idea of continuity at a point.

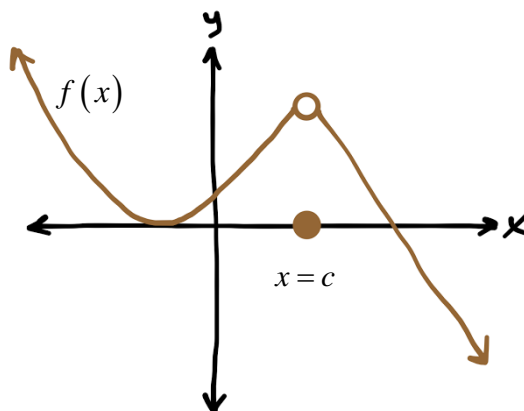


When looking for a **limit value** at  $x=c$ , imagine that you've got a thick vertical line covering up  $x=c$  with only the graph showing on either side of  $x=c$ . You are now looking to see what  $y$ -value(s) the graph is approaching on either side of  $x=c$ . If the graphs appear to be approaching the same  $y$ -value, the limit exists and is that  $y$ -value. Otherwise, the limit does not



When looking for a **function value** at  $x=c$ , imagine that you've got vertical blinders on either side of  $x=c$  with only the vertical sliver at  $x=c$  visible between them. You are now looking for the dot or the piece of the graph that exists in that narrow sliver. If it exists, the  $y$ -value of the dot is the function value  $f(c)$ .

It's only when you put the two together that you get the full picture of what the graph looks like at  $x=c$ .



**Example 13:**

Below is the graph of a function  $f(x)$ , whose scientific name is *Graphus Interruptus*. Evaluate the following:

(a)  $\lim_{x \rightarrow -2} f(x) =$

(b)  $\lim_{x \rightarrow 2^-} f(x) =$

(c)  $f(4) =$

(d)  $\lim_{x \rightarrow 4} f(x) =$

(e)  $\lim_{x \rightarrow 11^+} f(x) =$

(f)  $\lim_{x \rightarrow 5^-} f(x) =$

(g)  $f(2) =$

(h)  $\lim_{x \rightarrow 6^+} f(x) =$

(i)  $\lim_{x \rightarrow 6^-} f(x) =$

(j)  $\lim_{x \rightarrow 8} f(x) =$

(k)  $f(-2) =$

(l)  $\lim_{x \rightarrow 0^-} f(x) =$

