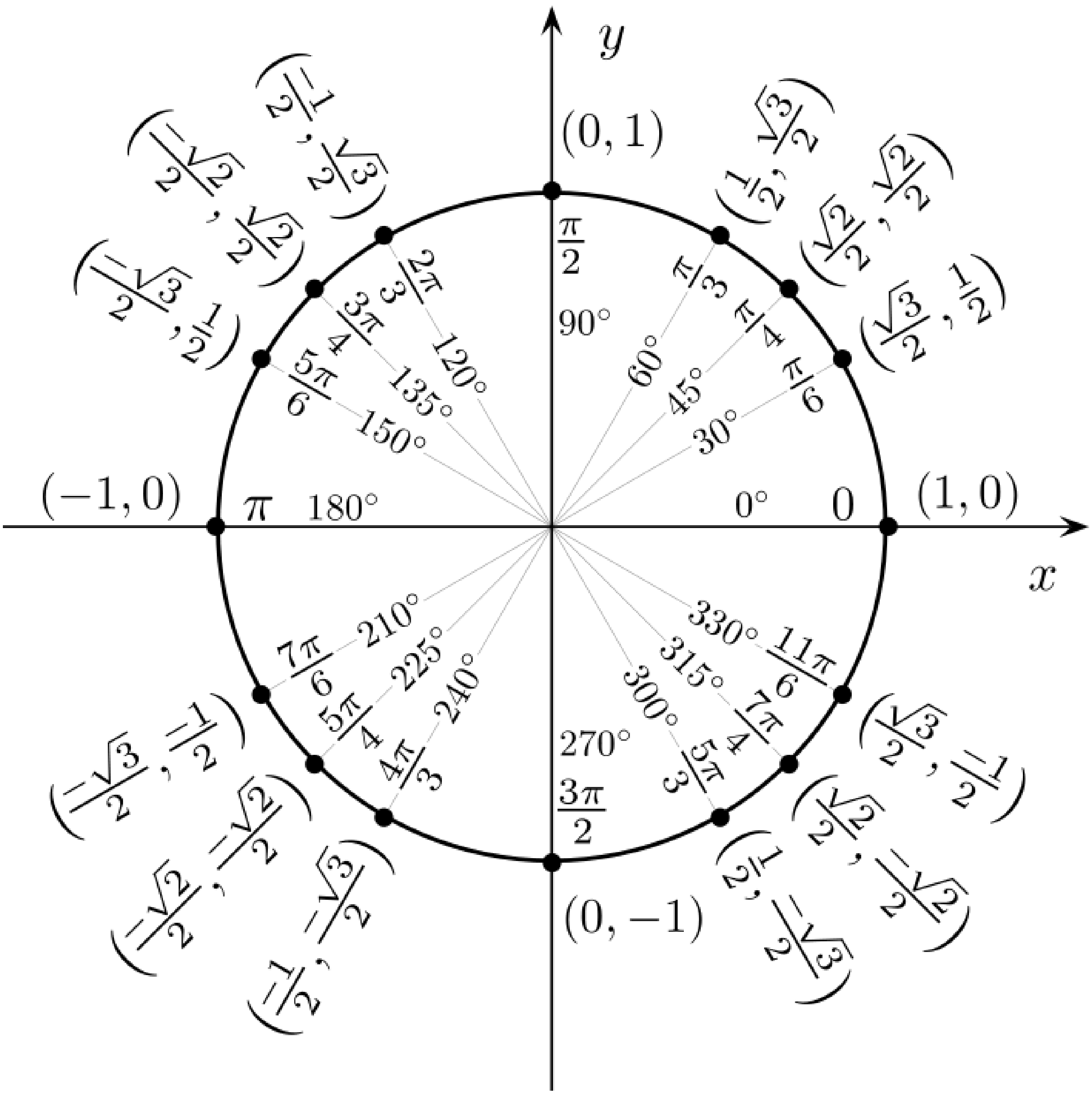
§P.7—Trigonometry

What's round and can cause major headaches?

The Unit Circle.

The Unit Circle will only cause you headaches if you don't know it. Using the Unit Circle in Calculus is equivalent to using your multiplication facts in Algebra. In case you've forgotten what it looks like, here it is.

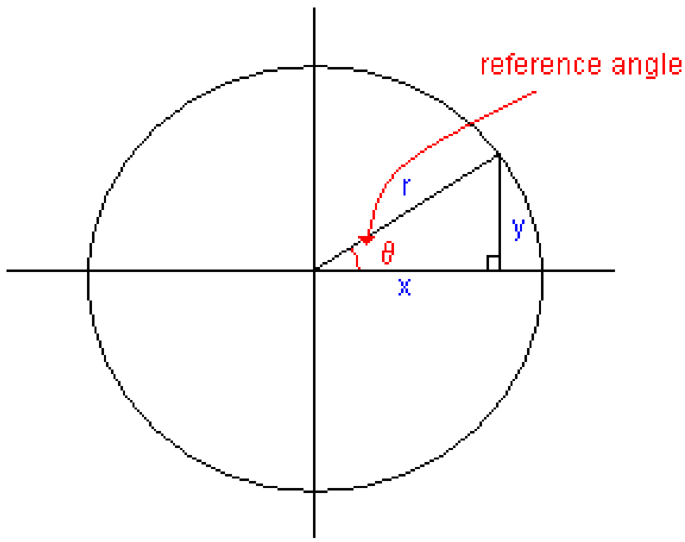


The ordered pairs on the Unit Circle give the cosine and sine values of some very useful angles in radians, such that $(x, y) = (\cos\theta, \sin\theta)$. Just knowing the cosine and sine values of these angles, with the help of the Reciprocal and Ratio Identities, you will be able to also find the other four trigonometric values, namely tangent, cotangent, secant, and cosecant.

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x} \qquad \tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

But what exactly ***is*** a trig ratio anyway?

You should never forget to remember that a trig ratio is nothing more than a unitless ratio of two of the three sides of a right triangle whose reference angle is determined by the independent angle θ.



The definition of the six trig ratios in terms of *x*, *y*, and *r* (which will work for a circle of any size, not just the unit circle when $r = 1$)

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Regardless of the size of the circle, these ratios, for given angles, will always simplify to the same value. This is because the scale factor always divides out (as well as any units). Test out your unit circle memory by quickly giving the simplified, exact value of the following:

a) $\sin\frac{5\pi}{3}$ b) $\cos\frac{5\pi}{6}$ c) $\tan\frac{7\pi}{4}$ d) $\cot\frac{3\pi}{2}$ e) $\sec\frac{3\pi}{4}$ f) $\csc\pi$

You should have correctly obtained the following:

$$\text{a) } \sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2} \qquad \text{b) } \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \qquad \text{c) } \tan\frac{7\pi}{4} = -1$$

$$\text{d) } \cot\frac{3\pi}{2} = 0 \qquad \text{e) } \sec\frac{3\pi}{4} = -\sqrt{2} \qquad \text{f) } \csc\pi = \text{DNE or Does Not Exist}$$

If you have a headache at this point, you should (re)memorize the Unit Circle and begin quizzing and re-quizzing yourself until you've mastered this.

Sometimes, though, we'll be working with angles and ratios not on the unit circle, and we'll might not even have our calculator.

Example 1:

Find all the remaining five trig values of θ if $\sin\theta = -\frac{3}{4}$ and $\cot\theta > 0$, Then find θ such that $0 \le \theta < 2\pi$. What is the reference angle? What other angle $0 \le \beta < 2\pi$ has the same sine value as θ?

It's worth noting here the difference between an inverse trig ***operation***, versus and inverse trig ***function***. When solving for an angle, we take the inverse (or arc) trig function of both sides of the equation. We must then find all the solutions in a specified interval, such that the trig function of each angle is the original ratio. For example . . .

Example 2:

For (a) $0 \le \theta < 2\pi$ and (b) all reall numbers, solve $\sin\theta = -\frac{\sqrt{2}}{2}$

In calculus, if the inverse or arc trig equation is explicitly given to us in the problem, we will assume it is talking about the inverse trig **function**. That means for any input (a ratio) there can only be **one output**.

Here's a similar type of question presented slightly differently.

Example 3:

Find an algebraic expression for $\tan\left(\arcsin t^2\right)$.

Another important trigonometric concept integral for calculus success is the ability to use trig identities to simplify expressions. We've already see the Reciprocal and Ratio Identities, but there are a handful more that show up from time to time.

As a general rule throughout calculus, we will want to simplify as early as we can in a problem, and as often as we can as we work through the problem. Things we can do are to expand expressions, combine like terms, divide out common factors, simplify exponents and log expressions, and finally, make trigonometric identity substitutions. Remember: **Simplify Early And Often!**

The Pythagorean Identities (PIDs):

Papa PID	**Baby PID 1**	**Baby PID 2**
$\cos^2 x + \sin^2 x = 1$	$1 + \tan^2 x = \sec^2 x$	$1 + \cot^2 x = \csc^2 x$

These identities can be algebraically manipulated to create other equivalent forms, for example:

$$\sec^2 x - \tan^2 x = 1$$

The Double Angle Identities:

Single Term	Difference of Squares
$\sin(2x) = 2\sin x \cos x$	$\cos(2x) = \cos^2 x - \sin^2 x$

These identities allow you to express any angle in terms of an angle half its size. Later in your studies, these identities will be useful to rewrite in integral expression into something more easily recognized. You'll see.

The Power-Reducing Identities:

Single Term	Difference of Squares
$\cos^2 x = \frac{1}{2}(1+\cos 2x)$	$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

The identities will also be invaluable when we study integration, as they allow us to go from staring at something that's impossible to do, to something that's as easy as taking candy away from a baby who's trying to give away his candy.

That's basically it. Know your identities, know the Unit Circle, and don't mess up.

Here' a cute Teddy Bear, poorly outlined, with a simple message for you.

Let's get on to calculus.