AP® Calculus AB
2015 Free-Response Questions

CALCULUS AB
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

1. The rate at which rainwater flows into a drainpipe is modeled by the function R, where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ $= y1$ cubic feet per hour, t is measured in hours, and $0 \le t \le 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ $= y2$ cubic feet per hour, for $0 \le t \le 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

   (a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \le t \le 8$?

   (b) Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.

   (c) At what time t, $0 \le t \le 8$, is the amount of water in the pipe at a minimum? Justify your answer.

   (d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

(a) $H_2O = \int_0^8 R(t)\,dt = 76.570 \text{ ft}^3$

(b) In rate @ $t = 3$hrs $= R(3) = 5.086 \text{ ft}^3/\text{hr}$

Out rate @ $t = 3$hrs $= D(3) = 5.4 \text{ ft}^3/\text{hr}$

Since $5.4 > 5.086$, the amount of water in the pipe is decreasing at $t = 3$ hrs.

(c) Let $W(t)$ = Water in pipe at time t. $W(t) = 30 + \int_0^t (R(x) - D(x))\,dx$

critical values: $W'(t) = 0 \rightarrow R(t) = D(t)$

$t = 3.271$ hrs $= A$ (min)

$W(0) = 30 \text{ ft}^3$

$W(8) = 30 + \int_0^8 (R(t) - D(t))\,dt = 48.543 \text{ ft}^3$

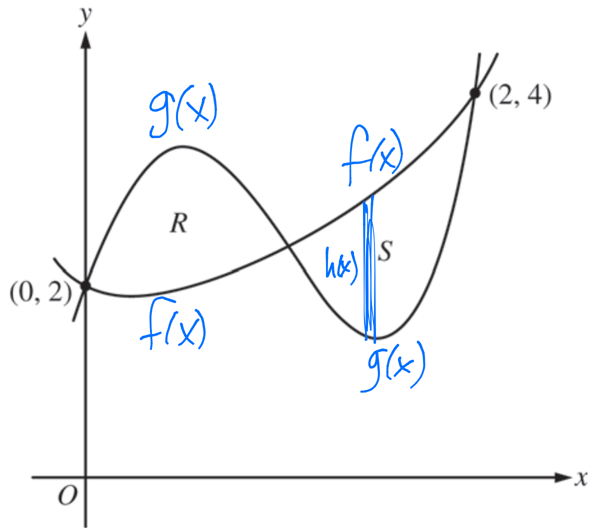
$W(A) = 30 + \int_0^A (R(t) - D(t))\,dt = 27.964 \text{ ft}^3$

Since $R(t)$ & $D(t)$ are continuous for $0 \le t \le 8$, the water in the pipe is a minimum at $t = 3.271$ hrs

*or water is a min at $t = 3.271$ hrs since $W'(t) < 0$ $\forall x \in [0, 3.271]$ hrs and $W'(t) > 0$ $\forall x \in [3.271, 8]$ hrs

(d) $W(t) = 50 \text{ ft}^3$

$50 = 30 + \int_0^t (R(x) - D(x))\,dx$



2. Let f and g be the functions defined by $f(x) = 1 + x + e^{x^2 - 2x}$ and $g(x) = x^4 - 6.5x^2 + 6x + 2$. Let R and S be the two regions enclosed by the graphs of f and g shown in the figure above.

(a) Find the sum of the areas of regions R and S.

(b) Region S is the base of a solid whose cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

(c) Let h be the vertical distance between the graphs of f and g in region S. Find the rate at which h changes with respect to x when $x = 1.8$.

(a) $f(x) = g(x)$

$x = 0,\ 1.032 = A,\ 2$

$\text{Sum} = \int_0^A (g(x) - f(x))\,dx + \int_A^2 (f(x) - g(x))\,dx$

$= 2.004$

(b) $\text{Volume} = 1 \cdot \int_A^2 (f(x) - g(x))^2\,dx$

$= 1.283$

END OF PART A OF SECTION II

(c) $h(x) = f(x) - g(x)$

$h'(x) = f'(x) - g'(x)$

$h'(1.8) = f'(1.8) - g'(1.8)$

$= -3.811$

CALCULUS AB

SECTION II, Part B

Time—60 minutes

Number of problems—4

No calculator is allowed for these problems.

12 8 4 16

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\,dt$ in the context of the problem.

Approximate the value of $\int_0^{40} |v(t)|\,dt$ using a right Riemann sum with the four subintervals indicated in the table.

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute. Find Bob's acceleration at time $t = 5$.

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

(a) $v'(16) \approx \dfrac{240-200}{20-12} = \dfrac{40}{8} = 5$ meters/min^2

(b) $\int_0^{40} |v(t)|\,dt$ is the total distance, in meters, Johanna travels from $t=0$ minutes to $t=40$ minutes.

$\int_0^{40} |v(t)|\,dt \approx (12)(200) + 8(240) + (4)(|-220|) + (16)(150)$ meters $= 7600$ meters

(c) $B'(t) = 3t^2 - 12t$

$B'(5) = 3(5^2) - 12(5)$ meters/min^2

$= 15$ meters/min^2

(d) Avg velocity $= \dfrac{\int_0^{10} B(t)\,dt}{10-0}$

$= \left(\frac{1}{10}\right)\left(\frac{1}{4}t^4 - 2t^3 + 300t\right)\Big|_0^{10}$

$= \left(\frac{1}{10}\right)\left[\left(\frac{10000}{4} - 2(1000) + 3000\right)\right]$ meters/min

$= 350$ meters/min

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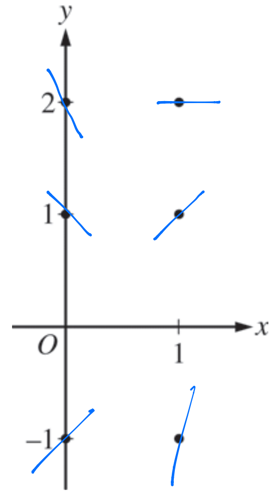
4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

(b) $\frac{dy}{dx} = 2x - y$

$\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx}$

$\frac{d^2y}{dx^2} = 2 - 2x + y$

In Quadrant II, $x<0, y>0$

So $\frac{d^2y}{dx^2} > 0$.

In qudrant II, the solution curves are concave up.

(c) At $f(2) = 3$

$\left.\frac{dy}{dx}\right|_{(2,3)} = (2)(2) - 3 = 1 \neq 0$

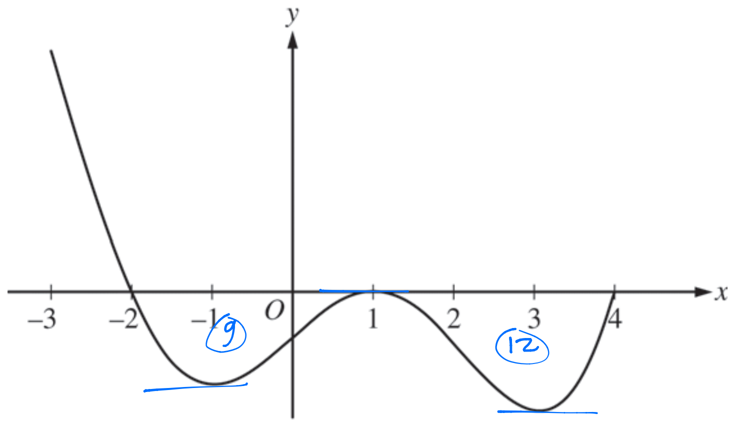
* So f does not have a critical value at (2,3), and so has Neither a relative min nor relative max at (2,3).

(d) for $y = mx+b$ to be a solution (linear), $\frac{d^2y}{dx^2} = 0$; from part (b)

$\frac{d^2y}{dx^2} = 2 - 2x + y = 0$

$y = 2x - 2$

So $m=2$, $b=-2$.



Graph of f'

5. The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x-axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.

   (a) Find all x-coordinates at which f has a relative maximum. Give a reason for your answer.

   (b) On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing? Give a reason for your answer.

   (c) Find the x-coordinates of all points of inflection for the graph of f. Give a reason for your answer.

   (d) Given that $f(1) = 3$, write an expression for $f(x)$ that involves an integral. Find $f(4)$ and $f(-2)$.

(a) $f'(x) = 0$
at $x = -2, 1, 4 \rightarrow$ critical values
* At $x = -2$, $f(x)$ has a relative max since $f'(x)$ changes from positive to negative at $x = -2$.

(b) f is concave down when f' is decreasing.
f is decreasing when f' is negative.
this happens for $-2 < x < -1$ and $1 < x < 3$.

(c) $f'' = 0$ when f' has horizontal tangents.
$x = -1, 1, 3 \rightarrow$ p.i.v.s

* f has inflection points at $x = -1$ and $x = 3$ since the slopes of f' change from negative to positive at $x = -1$ & $x = 3$.
* f also has an inflection point at $x = 1$ since the slopes of f' change from positive to negative at $x = 1$.

(d) $f(x) = 3 + \int_1^x f'(t)\,dt$

$f(4) = 3 + \int_1^4 f'(t)\,dt = 3 + (-12) = -9$

$f(2) = 3 + \int_1^{-2} f'(t)\,dt = 3 + 9 = 12$

6. Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\dfrac{dy}{dx} = \dfrac{y}{3y^2 - x}$.

(a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.

(b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.

(c) Evaluate $\dfrac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

(a) $y^3 - xy = 2$, $\dfrac{dy}{dx} = \dfrac{y}{3y^2 - x}$

$\left.\dfrac{dy}{dx}\right|_{(-1,1)} = \dfrac{1}{3+1} = \dfrac{1}{4} = m$

equation: $\boxed{y = 1 + \frac{1}{4}(x+1)}$

(b) vertical tangent line when $3y^2 - x = 0$

$x = 3y^2$

Subbing into 1st equation: $y^3 - (3y^2)y = 2$

$y^3 - 3y^3 = 2$

$-2y^3 = 2$

$y^3 = -1$

$y = -1$

So $x = 3(-1)^2 = 3$

* So vertical tangent line at point $(3,-1)$

(c) $\dfrac{dy}{dx} = \dfrac{y}{3y^2 - x}$

$\dfrac{d^2y}{dx^2} = \dfrac{(3y^2 - x)\frac{dy}{dx} - y\left(6y\frac{dy}{dx} - 1\right)}{(3y^2 - x)^2}$

from part (a) $\left.\dfrac{dy}{dx}\right|_{(-1,1)} = \dfrac{1}{4}$, so $\left.\dfrac{d^2y}{dx^2}\right|_{(-1,1)} = \dfrac{(3+1)(\frac{1}{4}) - (6(\frac{1}{4}) - 1)}{(3+1)^2}$

(walk away here!)

$= \dfrac{1 - \frac{1}{2}}{16}$

$= \dfrac{1}{32}$

STOP

END OF EXAM