§1.1b—Introduction to Continuity and Limits

Quick Review:

1. In general, what are the TWO situations you are looking for in the equation of a function that will limit its domain?

2. Find the domain of the following functions without your calculator. Write the domain in set notation or interval notation.

a. $f(x)=\dfrac{1}{x^2}$

b. $f(x)=\dfrac{x}{x^2+x-6}$

c. $f(x)=\sqrt{x}$

d. $f(x)=\sqrt{x-5}$

e. $f(x)=x\sqrt{4-3x}$

f. $f(x)=\dfrac{\sqrt{x}}{x^2-4}$

g. $f(x)=\dfrac{\sqrt{x+2}}{4-x}$

h. $f(x)=\dfrac{3\sqrt{x^2+7}}{x^2+5}$

i. $f(x)=\dfrac{4x+3}{\sqrt{3-x}}$

Continuity and introduction to limits:

For some of the functions above, the graphs do not exist anywhere to the left or to the right of a specific point, for instance, the graph of $f(x)=\sqrt{x}$ doesn't live anywhere in quadrants II and III. But for other functions, certain values that are not in the domain come in the middle of a graph, where the function values exist on either side. This means that there will be some "interruption" in our attempts to draw a smooth, connected graph from left to right. We will be forced to lift our pencil when we get to that particular *x*-value. Any such *x*-value that causes us to pick up our pencil is called a **discontinuity** of the graph. We are naturally interested in the behavior of the graph at these "forbidden" *x*-values.

In most of these cases, we won't be able to work the behavior out directly by direct substitution ... but we **can** see what it should be as you get closer and closer to those *x*-values!

Let's use this function as an example:

$$f(x) = \frac{x^2 - 1}{x - 1}$$

Notice $x = 1$ is not in the domain. Let's work it out for $x = 1$, that is, let's try to find $f(1)$.:

$$f(1) = \frac{1^2 - 1}{1 - 1} = \frac{1 - 1}{1 - 1} = \frac{0}{0}$$

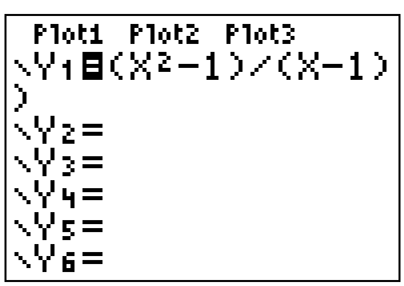
Now 0/0 is a difficulty! We don't really know the value of 0/0. It is one of many **Indeterminate Forms**. The function is undefined there, so we need another way of answering this. We DO know, however, that $x = 1$ is a discontinuity, since the graph exists on either side of it. I'll have to pick my pencil up to graph it, but what's actually going on at $x = 1$? Instead of trying to work it out for $x = 1$ let's try **approaching** it closer and closer, that is, let's "sneak up" on $x = 1$ from the left side of $x = 1$. We do this numerically by plugging in any *x*-values less than one (but not too less than one), then plugging in successive values that are closer and closer to one. As we do this, we look for a pattern in the resulting *y*-values.:

x	**(x^2-1)/(*x*-1)**
0.5	1.50000
0.9	1.90000
0.99	1.99000
0.999	1.99900
0.9999	1.99990
0.99999	1.99999
...	...

Being the great mathematical detectives that we are, we can now see that as *x* gets closer and closer to 1, then (x^2-1)/(*x*-1) gets closer and closer to 2

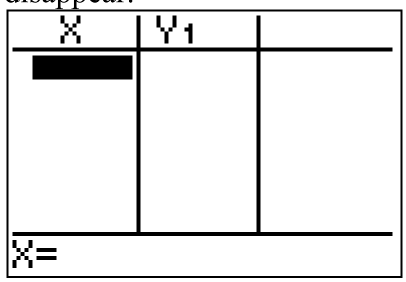
How do we do this on our calculators? Good question.

1. Put the function into "Y1="

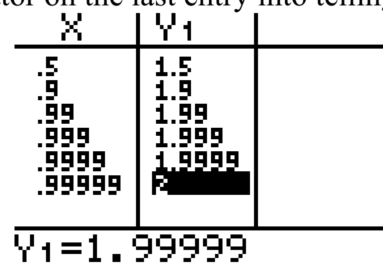


2. Go to "TBLSET" which is "2nd" "WINDODW" and select "Indpnt" to "ASK." It doesn't matter what the "TblStart" value is since you won't be using it.

3. Go to "TABLE" which is "2nd" "GRAPH." You should see a table ready to accept any x-value you care to input. If you've already done this, you may have values filled in, or if it is your first time, it might be blank. If you want to clear your table, simply click "DEL" with the cursor in the x-column and watch you values magically disappear.



4. Now type in values that you want and see what happens to the y-values. Are they approaching a specific value? Are they getting increasingly large? Here, we see they are approaching $y = 2$, in fact, we've "tricked" our calculator on the last entry into telling us the actual answer.



We are now faced with an interesting situation:

- When $x = 1$ we don't know the answer (it is **indeterminate**)
- But we can see that it is **going to be 2**

We want to give the answer "2" but can't, so instead mathematicians say exactly what is going on by using the special word "limit"

The **limit** of (x^2-1)/(x-1) as x approaches 1 is **2**

And it is written in symbols as:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

So the limit notation is a special way of saying, *"ignoring what happens when you get there, but as you get closer and closer the answer gets closer and closer to 2"*

Test Both Sides!

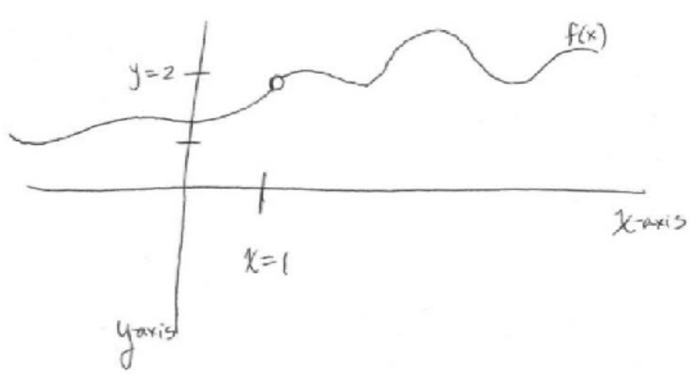
Finding the limit from one side, like we did above from the left of $x=1$ is like running up a hill and then finding the path **is magically "not there"...** but if you only check one side, who knows what happens? Does your running path stop altogether? Does it continue on along the same route on the other side of the hole? Does the path pick up somewhere else? All we really know is $\lim_{x\to 1^-} f(x)=2$. The negative sign that looks like an "ionic charge" means from the "negative side" or "left side" of $x=1$.

So we need to test it **from both directions** to be sure where it "should be." In other words, we need to find $\lim_{x\to 1^+} f(x)$, what the values are approaching from the "positive side" or "right side," of $x=1$:

x	**(x^2-1)/(*x*-1)**
1.5	2.50000
1.1	2.10000
1.01	2.01000
1.001	2.00100
1.0001	2.00010
1.00001	2.00001
...	...

We can see that the values are also heading for 2, so that's OK. In this case, since $\lim_{x\to 1^-} f(x)=2$ and $\lim_{x\to 1^+} f(x)=2$, we can say that the limit as *x* approaches 1 is 2. Mathematically, we simply write the ordinary limit as $\lim_{x\to 1} f(x)=2$. When writing this new notation without the small "+" or "—" signs, we mean that as *x* approaches 1 from **BOTH** sides, the *y*-values are approaching 2. Remember in this case, $f(1)$ is still undefined. If the two one-sided limits either didn't exist or were different values, we'd say the ordinary limit would not exist. In our case, they both equaled 2, so the ordinary limit existed and was also equal to 2.

So what do we really know about our function at $x=1$? We know the function doesn't exist there, but that the graph comes in to $y=2$ from both sides of $x=1$. If we were to draw this information, it might look something like this:



When this type of situation occurs, we say that a function has a **Point Discontinuity** at that value. In this case, we say that $f(x)$ has hole at the coordinate $(1,2)$. This would be analogous to running along the path, and getting to a point where there is a small, uncovered manhole in our way. It wouldn't take much to step over the small hole and keep going on the other side. All we would have to do to make the graph continuous again at that point, and the path safer for everyone, is to place a tiny "manhole cover" down right there, cover the hole, and the graph would again be smoooooooooooth. This is precisely why a point discontinuity is considered to be a **removable discontinuity** or **RD**.

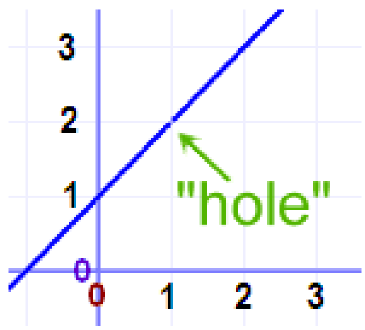
Let's take a closer look algebraically. We can simplify the original function the following way.

$$f(x)=\frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{x-1}=x+1,\ x\neq 1$$

This means the original function and the line $x+1$ look EXACTLY the same, except at the point $(1,2)$ where the line has a function value and $f(x)$ has a hole. Knowing this, we can use the equation of the line at $x=1$ to find the coordinate of the hole of $f(x)$.

At $x=1$, $x+1$ becomes $1+1=2$

The graph of $f(x)$ actually looks like this:



You graphing calculator will likely not show it unless you graph it in a "ZOOM" "Decimal" window.

So, in truth, we **cannot say what the value at *x*=1 is,** but we **can** say that as we approach 1, the **limit** is 2**.**

Is there a way to tell that this discontinuity was a removable point discontinuity simply from the equation? Well, yes and no. Here's a couple of things that are not absolute, but are true most of the time.

Remember above that we were able to "remove" the factor of $(x-1)$ from the denominator? Well, this is the factor that gave us the zero in the denominator. Since we were able to algebraically "remove" him from the function, he will (likely) be a "removable" point discontinuity.

Another way to look at it involves direct substitution. When we did this, we got $\frac{0}{0}$, in general, when this happens, you can count on having a $\frac{{}^{H}0}{0_{LE}}$ at that *x*-value. Hey! It works . . . most of the time, as you will see later.

Let's analyze a slightly different function now:

$$g(x) = \frac{x^2+1}{x-1}$$

Aside from having a different name than our last function, the only difference is that we are ADDING the terms in the numerator instead of subtracting. Will that change our domain????????

NO! We still cannot divide by zero, and the only value that causes this is still $x = 1$. So $D_g : \{x | x \neq 1\}$.

Naturally, we experiment by plugging in this "taboo" value: $g(1) = \frac{1^2+1}{1-1} = \frac{2}{0}$

Because of the zero in the denominator (as we suspected) the function does not exist at $x = 1$. But notice that we did NOT get $\frac{0}{0}$, instead we got $\frac{\neq 0}{0}$. Does that mean anything???? Well, it means that the discontinuity will likely NOT be a hole. So what will it be then? Let's use the limit to analyze the behavior of the *y*-values in the vicinity of $x = 1$. We'll use our calculators again.

From the left of $x = 1$

X	Y2	
.9	-18.1	
.99	-198	
.999	-1998	
.9999	-19998	
.99999	-2E5	

X=

It appears that the *y*-values are getting increasingly negative and NOT approaching a specific value, we can then say:

$$\lim_{x \to 1^-} g(x) = -\infty$$

By saying this, we're saying 2 things:

1. The limit does not exist (infinity has no limit)
2. The *y*-values are going down forever

Note that $-2E5 = -2 \times 10^5 = -200{,}000$

From the right of $x = 1$

X	Y2	
1.1	22.1	
1.01	202.01	
1.001	2002	
1.0001	20002	
	200002	

X=1.00001

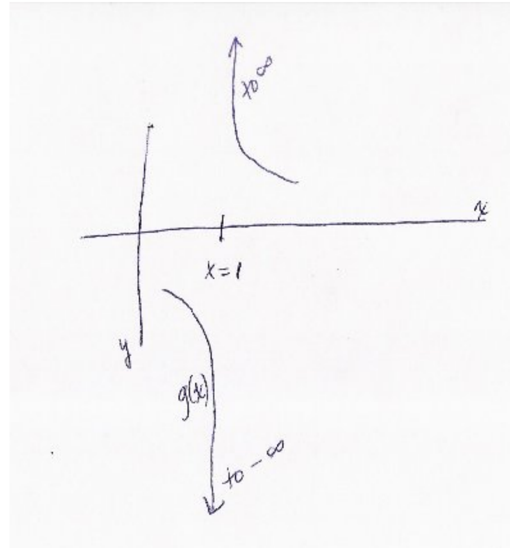
It appears that the *y*-values are getting increasingly positive and NOT approaching a specific value, we can then say:

$$\lim_{x \to 1^+} g(x) = \infty$$

In this case, we must say that the ordinary limit "does not exist" since we were approaching two different types of infinity on either side. If both sides were approaching positive infinity, we could use it for the ordinary limit, same goes for negative infinity.

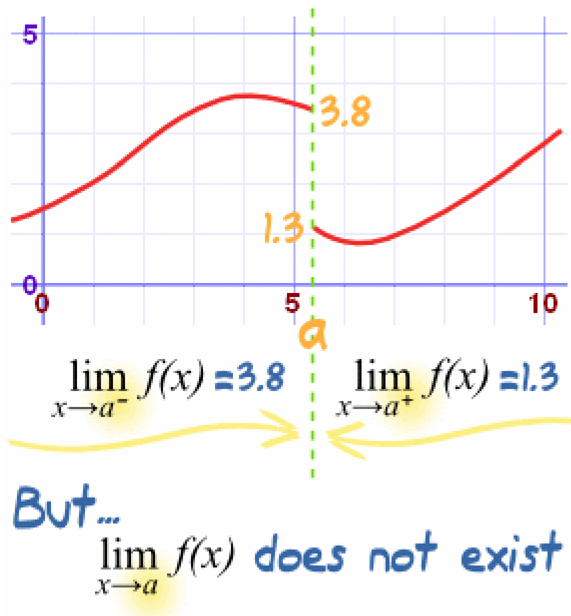
So what do we know about the function at $x = 1$?

We know enough to sketch something like the graph at right.



When it is different from different sides

What if we have a function $f(x)$ with a "break" in it like this:



This is a function where **the limit does not exist** at $x = a$ ... !

You can't say what it is, because there are two competing answers:

- 3.8 from the left, and
- 1.3 from the right

But again, we **can** use the special "-" or "+" signs (as shown) to define one sided limits:

- the **left-hand** limit (-) is 3.8
- the **right-hand** limit (+) is 1.3

And the ordinary limit **"does not exist"**

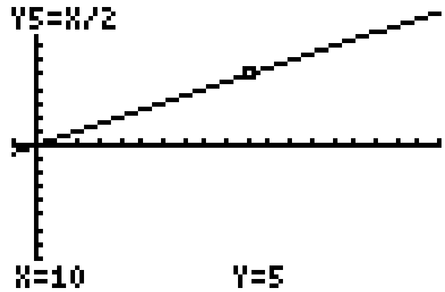
Notice in this situation, both of the one-sided limits STILL exist, but because they're different, we must conclude the ordinary limit does not exist (**DNE**). This would be analogous to running along the path, then getting to the point where the path "magically" picked back up somewhere else, hence the **Jump Discontinuity**. We would NOT be able to put a single "manhole cover" down anywhere at $x = a$ to make the paths connect or to make the graph smooth and continuous. This is exactly why a jump discontinuity is a **non-removable discontinuity**.

Are limits only for difficult functions at "forbidden" values?

Limits can be used even if you **know the value when you get there**! Nobody said they are only for difficult functions.

For example:

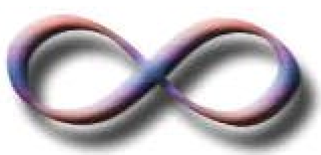
$$\lim_{x\to 10}\frac{x}{2}=5$$



We know perfectly well that 10/2 = 5, this is precisely the function value there, but limits can still be used. This would be analogous to running along the path at any point where there is no hole, no jump, no chasm, no canyon, etc. You're simply running along normally.

In this case, the limit and the function value both exist and are equal at that point. When this happens, we can say with confidence that the function is **continuous** at that point.

Approaching Infinity



Infinity is a very special idea. We know we can't reach it, but we can still try to work out the value of functions that have infinity in them.

Let's start with an interesting example.

Question: What is the value of $\frac{1}{\infty}$?

Answer: We don't quite know!

Why don't we know?

The simplest reason is that ***Infinity*** is not a number, it is an idea. So $\frac{1}{\infty}$ is a bit like saying $\frac{1}{beauty}$ or $\frac{1}{tall}$.

Maybe we could simply say that $\frac{1}{\infty}=0$, ... but that is a problem too, because if we divide 1 into infinite pieces and they end up 0 each, what happened to the 1?

In fact $\frac{1}{\infty}$ is known to be **undefined** . . . . .

. . . . But We Can Approach It!

So instead of trying to work it out for infinity (because we can't get a sensible answer), let's try larger and larger values of x for $f(x)=\dfrac{1}{x}$

x	$1/x$
1	1.00000
2	0.50000
4	0.25000
10	0.10000
100	0.01000
1,000	0.00100
10,000	0.00010

Now we can see that as x gets larger, $1/x$ tends towards 0. We are now faced with an interesting situation:

- We can't say what happens when x **gets to** infinity (because it won't!)
- But we can see that $1/x$ is going **towards** 0

We want to say that $f(\infty)=\dfrac{1}{\infty}=0$ but we **can't**, so instead mathematicians say exactly what is going on by using that special word again: "limit"

The **limit** as x approaches Infinity of $1/x$ is **0**

And write it like this:

$$\lim_{x\to\infty}\frac{1}{x}=0$$

In other words:
As x approaches infinity and gets infinitely large, then $1/x$ gets infinitely small approaching 0

The limit is a Notion of Motion, so when you see "limit**", think "**approaching**"

End Behavior

Whenever we look at what's happening to a graph of a function $f(x)$ for very large values of x in either the positive or negative direction, we are looking at its **<u>End Behavior</u>**.

That is, when we find either $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$ we are looking to see what the behavior of the graph is out at its left and right ends, even thought the graphs may never actually end.

What are some of the behaviors we can expect to see?

1. <u>The graph can fail to exist</u>: For instance, $\lim_{x\to-\infty}\sqrt{x}=DNE$ since the graph doesn't live to the left of the y-axis. It doesn't even make sense to talk about the left-end behavior of a function like this.

2. <u>The graph can increase without bound</u>: For instance, $\lim_{x\to\infty}\sqrt{x}=\infty$. Notice here that we are saying the limit EQUALS infinity, even though infinity is not a number. The limit actually does not exist,

but by writing it this way, we are saying that as x approaches infinitely large positive values, the function will continue to get infinitely large in the positive direction, unbounded.

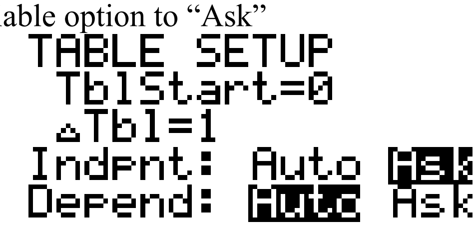
3. <u>The graph can decrease without bound</u>: For instance, $\lim_{x\to\infty}\left(-2x+1\right)=-\infty$. The limit still does NOT exist, since negative infinity is not a number, be we give great information by saying $-\infty$. We are simply saying that on the graph of this negatively sloping line, as we move out on the graph towards the right forever and ever, we will continue downward forever and ever.

4. <u>The graph can taper off and approach a specific *y*-value</u>: For instance, our example above, $\lim_{x\to\infty}\frac{1}{x}=0$. When this happens, we say there is a **<u>Horizontal Asymptote</u>** or HA at that specific *y*-value. Note: A Horizontal Asymptote is NOT a type of discontinuity!!!!! It does NOT force us to lift our pencil when we draw the function from left to right. In fact, as you will see later in the year, we can cross an HA anywhere on a graph, since the HA only describes where the *y*-values will EVENTUALLY taper off to somewhere WAY down the graph. The **<u>Vertical Asymptote</u>** is the only type of asymptote that is a discontinuity, since it forces us to lift our pencil.

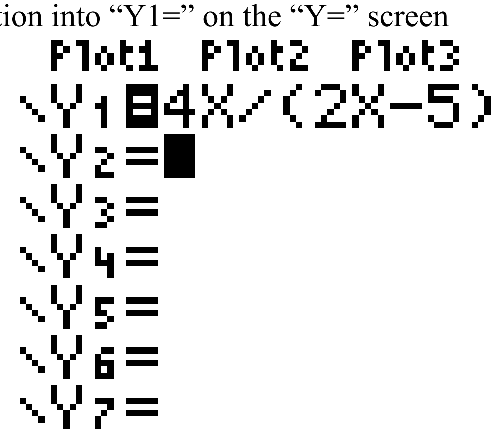
You can use your graphing calculator to find limits at infinity (You can also use this method to find limits on both sides of specific *x*-values.) I'll walk you through it for the following function: $f(x)=\frac{4x}{2x-5}$.

Here's what you'll need to do:

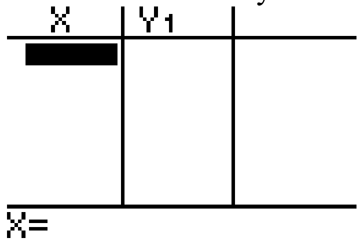
1. Make sure your batteries in your calculator are energized
2. Turn your calculator on
3. Press "2nd" "WINDOW" to access the "Table Setup" screen
4. Select your Independent variable option to "Ask"



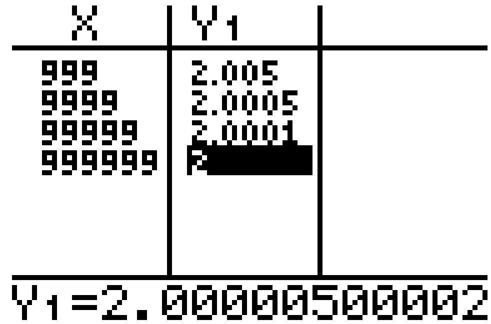
5. Type your desired function into "Y1=" on the "Y=" screen



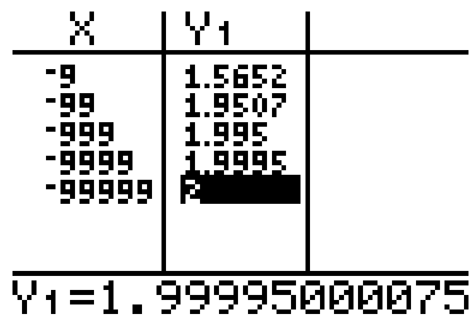
6. Press "2nd" "GRAPH" to access the table. It will likely be blank, but if it's not, that's OK.



7. Type in increasing large values of x into the first column and press "ENTER." After a while, being the good mathematical detective, you should be able to figure out which of the above 4 cases of end behavior the graph exhibits. Moreover, if there IS a horizontal asymptote, you should be able to figure out its y-value. Notice that on the 4th row, the calculator displays a "2" in the column, but when you cursor over it, you see the decimals that would not fit in the narrow column. In this case, I'm going to say that $\lim_{x\to\infty} f(x) = 2$ and that there is a HA at $y = 2$.

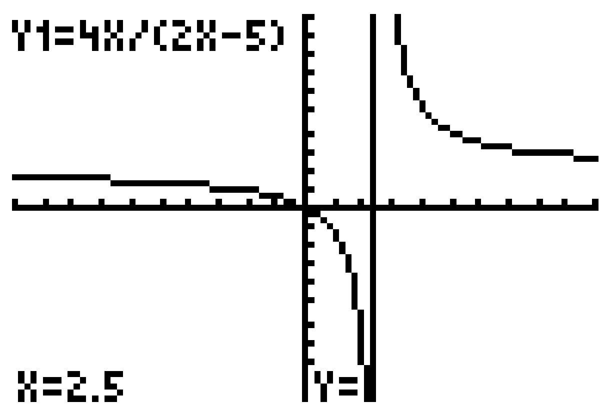


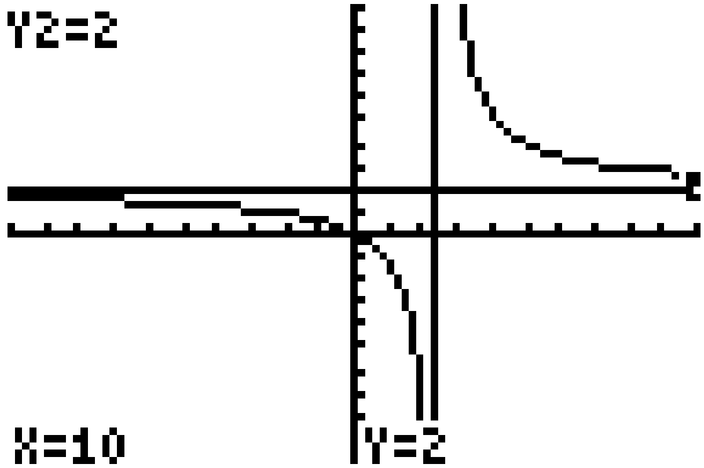
8. Repeat the process for negatively huge x-values. I'm going to go out on a limb in this case and say that $\lim_{x\to-\infty} f(x) = 2$ as well. We already knew about the HA at $y = 2$, but now we know the graph tapers off to this y-value at both ends.



Here's what the graph looks like in a standard viewing window:

Notice that the value of $x = \frac{5}{2} = 2.5$ is not in the domain. There is a Non-removable, Infinite discontinuity there, also called a Vertical Asymptote or VA. But also notice how the graph seems to level off on each end. Guess to what y-value they level? Yep, $y = 2$, the value of the HA. In the second graph, I have actually graphed the horizontal line, $y = 2$ so you can "see" the HA, like you can "see" the VA.




Y2=2
X=10
Y=2