§5.2—Slope Fields

Many times, differential equations are neither separable nor solvable by analytic methods. In such a case, we are still interested in the general and particular solutions. How can we do this? Is it even possible? Fear not, there IS a way to *graphically* solve such (and any) differential equation and get a nice visual of the solution curves.

Definition:

A **Slope Field** is a graphical general solution to a differential equation.

A **Particular Solution** in a Slope Field is a ***Continuous*** ***Function*** passing through an initial condition.

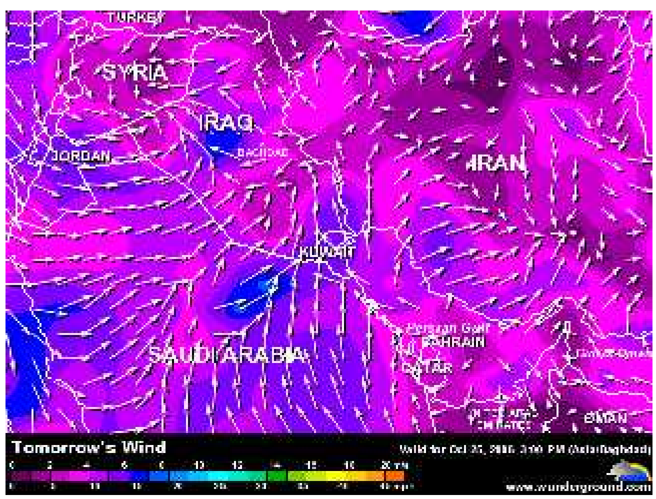
Recall that a derivative function gives us the slopes of a function at a point. When we work with differential equations, we are dealing with expressions in which the derivative appears as a variable. For example, we might be asked to analyze the differential equation:

$$\frac{dy}{dx} = x^2$$

If we replace the $\frac{dy}{dx}$ in the above equation with what it represents, we get the following statement:

$$\text{slope at any point } (x, y) = x^2$$

It is possible to learn a great deal about a differential equation, even when we don't know how (or deliberately pretend that we don't know how) to solve the equation, by looking at a picture of its trajectory space or **Slope Field**.



In particular, a solution is unique if no two curves in a trajectory space, or slope field intersect. Imagine several functions whose only difference is a different C value, so that all the graphs are "parallel" and differ only by a vertical shift.

It is actually quite easy to visualize a trajectory space (henceforth exclusively referred to by "**Slope Field**" and as an occasional "**Vector Field**") by sketching the slope field (sometimes also called a "flow field" or "direction field") of the differential equation.

In general, the slope field of the differential equation $y = \frac{dy}{dx}$ is defined in the following manner:

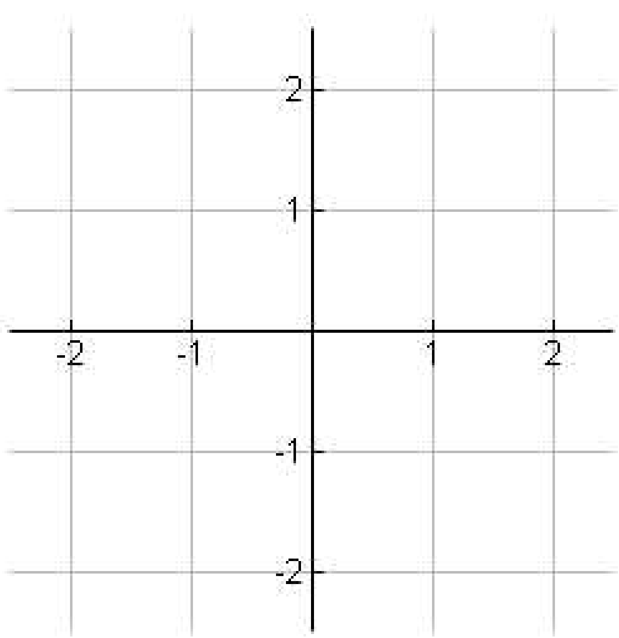
> To every point (x, y) in the domain of *f*, assign a small piece of a tangent line with a slope of $f'(x, y) = \left.\frac{dy}{dx}\right|_{(x,y)}$. The slope field at that point will be a small piece of a tangent line with a slope of $f'(x, y)$ tangent to a curve that passes through the point (x, y).

For our example, we're seeking a function whose slope at any point in the (x, y)-plane is equal to the value of x^2 at that point. Let's illustrate that by examining a few selected points.

At the point $(1,2)$, the slope would be $1^2 = 1$.
At the point $(5,3)$, the slope would be $5^2 = 25$.
At the point $(-3,11)$, the slope would be $(-3)^2 = 9$.

It would be impossible to produce a slope field covering the entire, infinite, Cartesian plane. Instead, for our example, let's restrict the section of the plane we consider to the following intervals: $x \in [-2, 2]$ and $y \in [-2, 2]$ using the integer coordinates. Let's try to make one:

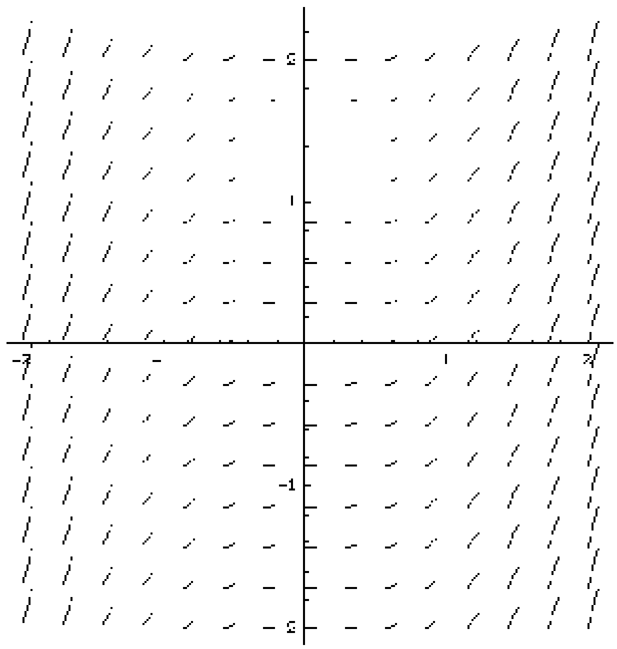
Example 1:
Draw a slope field for the differential equation: $\frac{dy}{dx} = x^2$ at the indicated points.



*Note: When drawing a piece of the tangent line at a point, draw the line long enough to see, but not so long that it interferes with the other tangent lines.
**Note: Be sure your slopes of 0, 1, -1, and ∞ are spot on. All other slopes must be at a steepness relative to these slopes and the others around it.

WHEN DRAWING A PARTICULAR SOLUTION TO A SLOPE FIELD, DRAW A ***CONTINOUS, FUNCTIONAL **CURVE** MAKING SURE TO PASS THROUGH THE GIVEN POINT.

If we continue doing this for many more points and plot a small piece of a tangent line at the give point having the calculated slope, we may get a picture that may look a little like this:



A sample slope field made with Mathematica

As previously mentioned, sometimes we don't know the formula for the solution to a differential equation. In cases such as this, a slope field is the perfect way to at least get a visual for what the general and particular solutions will look like.

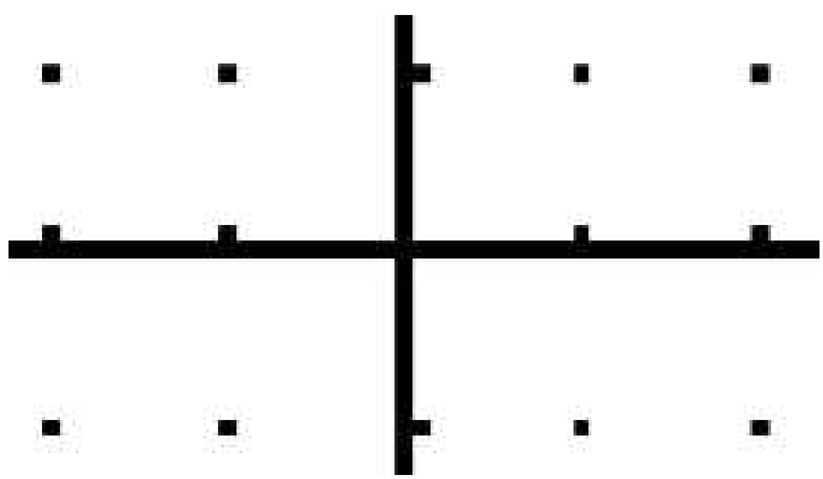
Example 2:
Sketch the graph of the solution for the **non-separable** initial value problems:

$y' = x + y$

(a) $y(0) = 1$

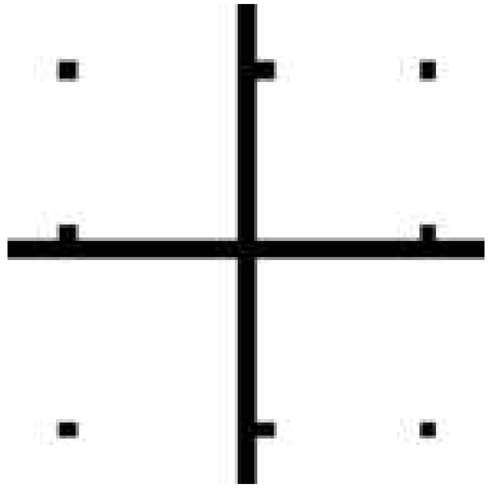
(b) $y(-1.5) = -0.5$

(c) $y(-1) = 0$

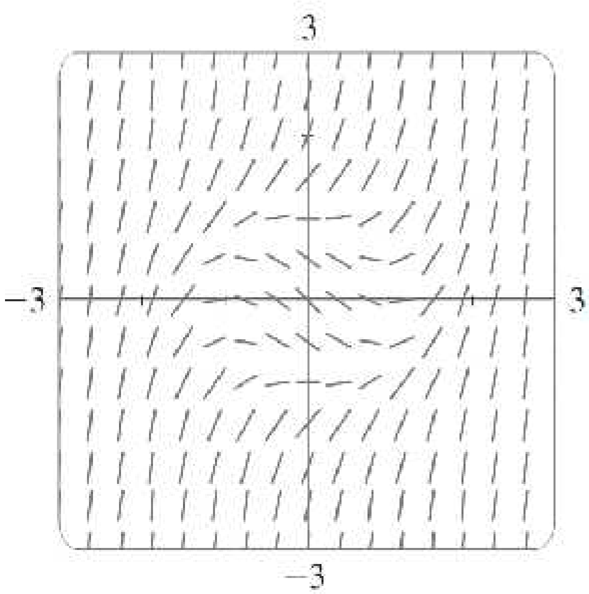


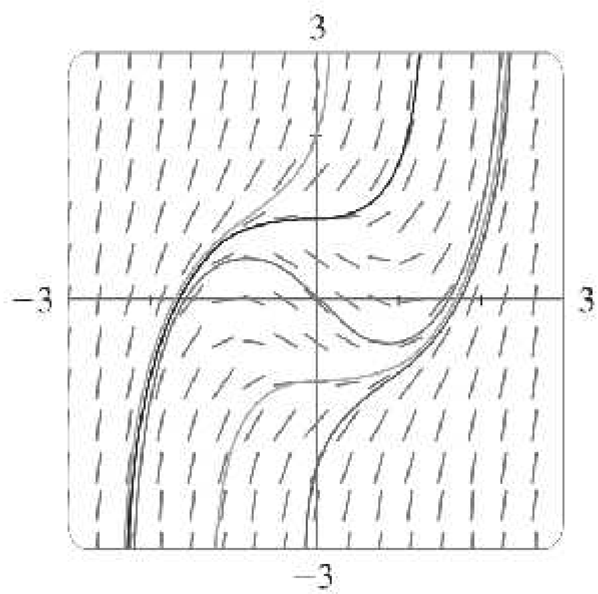
Example 3:

Sketch the slope field for the differential equation $\frac{dy}{dx} = x^2 + y^2 - 1$, then sketch the solution curve that passes through the origin.



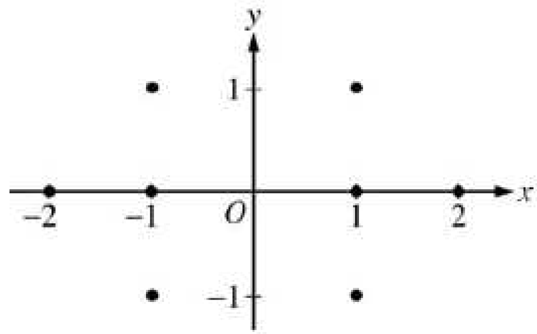
The more tangent line pieces we draw in a slope field, the better the picture of the solutions. In fact, if our line segments were so small that they were indistinguishable from points, we'd have the actual graph or the solutions. It is, however, very tedious and undesirable to compute and draw all those by hand. Our graphing calculators and computers are well-designed to do this. Below is a slope field of the previous problem done with a computer. Notice how we can now, with more confidence and accuracy, draw particular solutions as **<u>CONTINUOUS, FUNCTIONAL CURVES</u>**, such as those passing through $(0,-2)$, $(0,-1)$, $(0,0)$, $(0,1)$, and $(0,2)$.





Example 4:

For $\frac{dy}{dx} = \frac{y+2}{x}$, where $x \neq 0$,



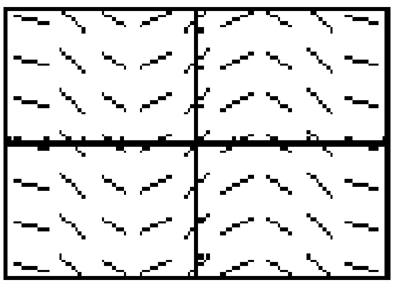
(a) On the axes provided, sketch a slope field for the given differential equation at the eight points indicated.

(b) Sketch the solution curve that passes through the point $(1,1)$.

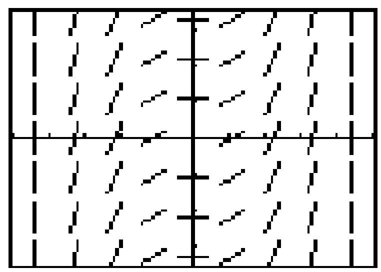
Example 5:

Match each slope field with the equation of the general solution it could represent.

I.



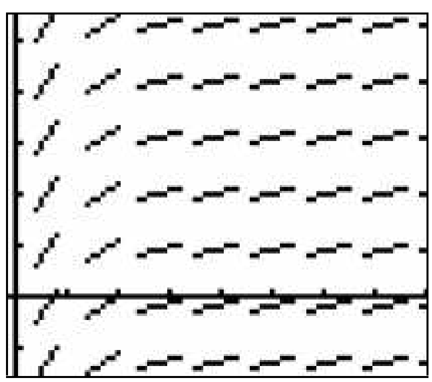
II.



(A) $y=1$ (B) $y=x$ (C) $y=x^2$ (D) $y=\frac{x^3}{6}$ (E) $y=\frac{1}{x^2}$ (F) $y=\sin x$

(G) $y=\cos x$ (H) $y=\ln|x|$

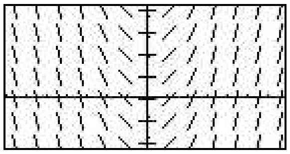
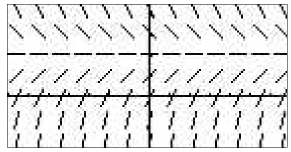
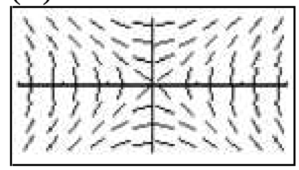
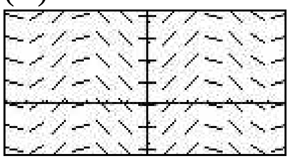
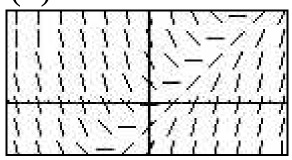
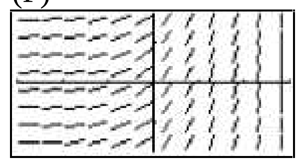
Example 6:



The slope field from a certain differential equation is shown above. Which of the following could be a specific solution to that differential equation?

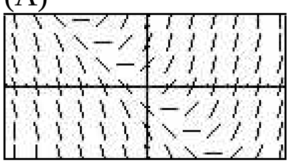
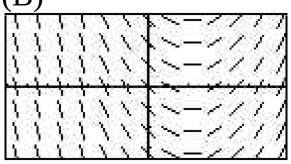
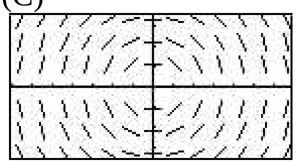
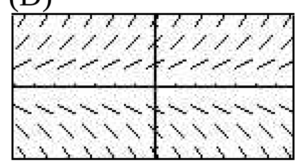
(A) $y=x^2$ (B) $y=e^x$ (C) $y=e^{-x}$ (D) $y=\cos x$ (E) $y=\ln x$

Example 7:
Match the slope fields with their differential equations.

(A)

(B)

(C)

(D)

(E)

(F)


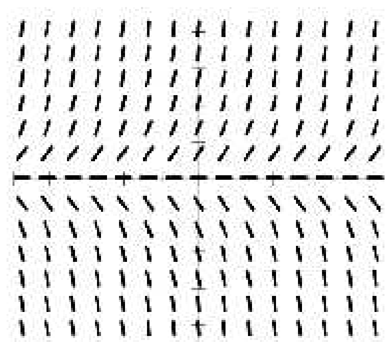
I. $\frac{dy}{dx} = e^x$ II. $\frac{dy}{dx} = \frac{x}{y}$ III. $\frac{dy}{dx} = 2 - y$ IV. $\frac{dy}{dx} = x$ V. $\frac{dy}{dx} = x - y$ VI. $\frac{dy}{dx} = \sin x$

Example 8:
Match the slope fields with their differential equations.

(A)

(B)

(C)

(D)


I. $\frac{dy}{dx} = .5x - 1$ II. $\frac{dy}{dx} = .5y$ III. $\frac{dy}{dx} = -\frac{x}{y}$ IV. $\frac{dy}{dx} = x + y$

Example 9:
The slope field for a differential equation is shown at right. Which statement is true for all solutions of the differential equation?



I. For $x < 0$, all solutions are decreasing.
II. All solutions level off near the *x*-axis.
III. For $y > 0$, all solutions are increasing

(A) I only (B) II only (C) III only
(D) II and III only (E) I, II, and III

Example 10:

The slope field for the differential equation $\frac{dy}{dx} = \frac{x^2 y + y^2}{4x + 2y}$ will have vertical segments when

(A) $y = 2x$ (B) $y = -2x$ (C) $y = -x^2$ only (D) $y = 0$ only (E) $y = 0$ or $y = -x^2$

Example 11:

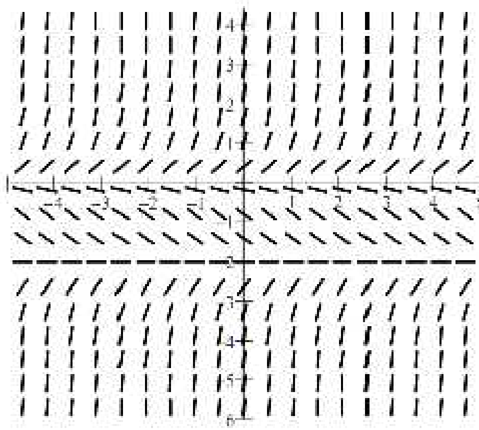
Which statement is true about the solutions, $y(x)$, of a differential equation whose slope field is shown at right?

I. If $y(0) > 0$, then $\lim_{x\to\infty} y(x) \approx 0$

II. If $-2 < y(0) < 0$, then $\lim_{x\to\infty} y(x) \approx -2$

III. If $y(0) < -2$, then $\lim_{x\to\infty} y(x) \approx 0$

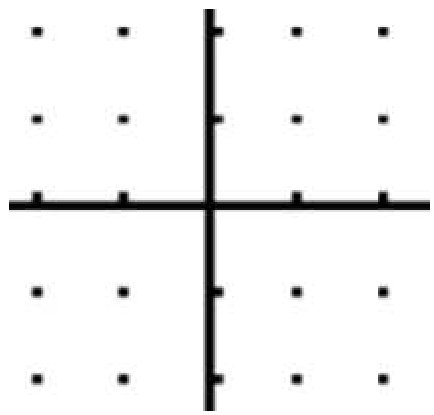
(A) I only (B) II only (C) III only
(D) II and III only (E) I, II, and III



Example 12:

Consider the differential equation given by $\frac{dy}{dx} = \frac{xy}{2}$.

(a) On the axes provided, sketch a slope field for the given differential equation.



(b) Let f be the function that satisfies the given differential equation. Write an equation for the tangent line to the curve $y = f(x)$ through the point (1, 1). Then use your tangent line equation to estimate the value of $f(1.2)$

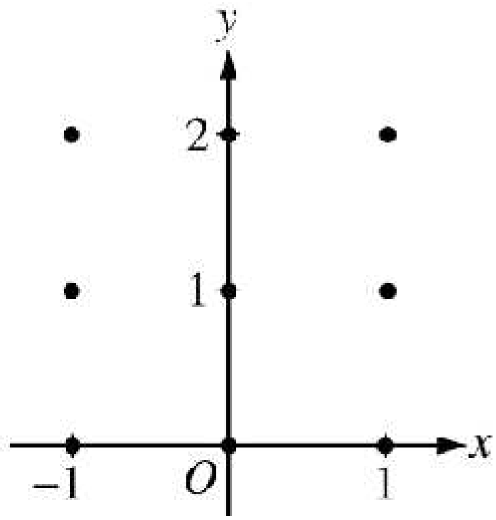
(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(1) = 1$. Use your solution to find $f(1.2)$.

(d) Compare your estimate of $f(1.2)$ found in part (b) to the actual value of $f(1.2)$ found in part (c). Was your estimate from part (b) an underestimate or an overestimate? Use your slope field to explain why.

Example 13:

AP 2007B-5 Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}x + y - 1$.

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.



(b) Find $\frac{d^2y}{dx^2}$ in terms of *x* and *y*. Describe the region in the *xy*-plane in which all solution curves to the differential equation are concave up.

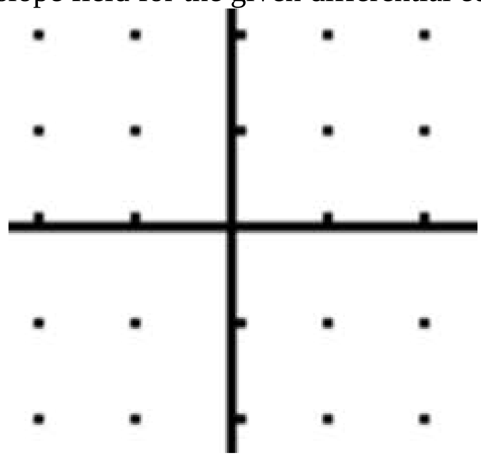
(c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = 1$. Does *f* have a relative minimum, a relative maximum, or neither at $x = 0$? Justify your answer.

(d) Find the values of the constants *m* and *b*, for which $y = mx + b$ is a solution to the differential equation.

Example 14:

Consider the differential equation given by $\frac{dy}{dx} = \frac{x}{y}$.

(a) On the axes provided, sketch a slope field for the given differential equation.



(b) Sketch a solution curve that passes through the point (0, 1) on your slope field.

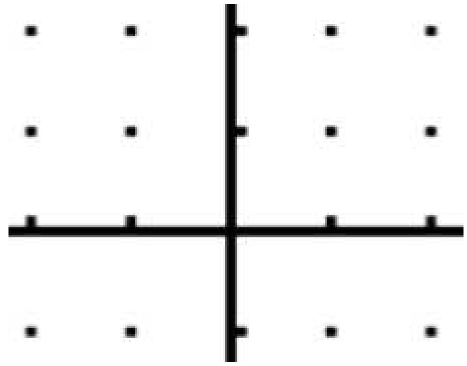
(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(0) = 1$.

(d) Sketch a solution curve that passes through the point $(0, -1)$ on your slope field.

(e) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(0) = -1$.

Example 15:

Consider the differential equation given by $\frac{dy}{dx} = \frac{2x}{x^2+1}$.



(a) On the axes provided, sketch a slope field for the given differential equation.

(b) Sketch a solution curve that passes through the point (0, 1) on your slope field.

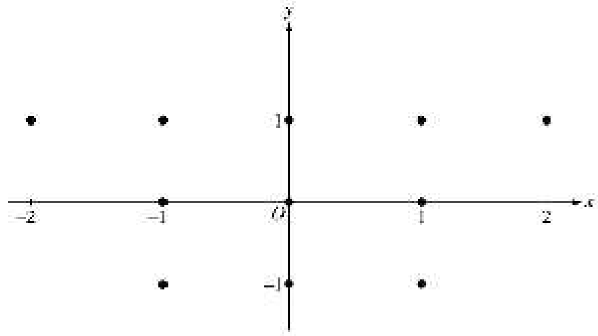
(c) Find the particular solution for your curve from part (b)

(d) Find $\frac{d^2y}{dx^2}$. For what values of x is the graph of the solution $y = f(x)$ concave up? Concave down?

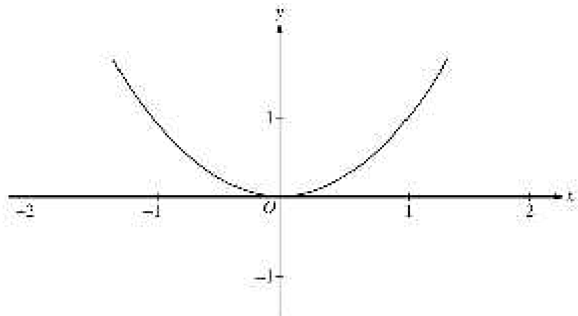
Example 16:
2000—BC6

Consider the differential equation given by $\frac{dy}{dx} = x(y-1)^2$.

(a) On the axes provided, sketch a slope field for the given differential equation at the eleven points indicated.



(b) Use slope field for the given differential equation to explain why a solution could not have the graph shown below.



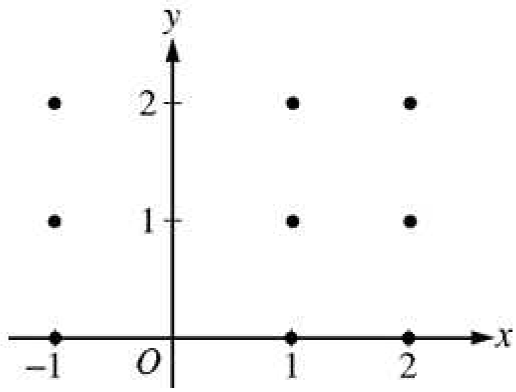
(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -1$.

(d) Find the range of the solution found in part (c).

Example 17:
AP 2008-5

Consider the differential equation $\frac{dy}{dx}=\frac{y-1}{x^2}$, where $x \neq 0$.

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.



(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 0$.

(c) For the particular solution $y = f(x)$ described in part (b), find $\lim_{x\to\infty} f(x)$.