§2.5—Rates of Change and Particle Motion I

If $f(x)$ represents a quantity, then $f'(x)$ or $\frac{df}{dx}$ represents the instantaneous rate of change of that quantity. As we have seen, $f(x)$ may describe a particle's position or its velocity, but $f(x)$ can represent ANY quantity, such as the area of a circle, the outdoor temperature, the amount of rainfall, number of people infected with a disease, etc. The derivative gives us a way to discuss how any of these quantities change, usually with respect to time, at a given moment.

We will start be verbally interpreting mathematical sentences involving the derivative of some function.

Example 1:
The temperature *T*, in degrees Fahrenheit, of a cold yam placed in a hot oven is given by $T = f(t)$, where *t* is the time in minutes since the yam was put in the oven.

(a) In a complete sentence with units, translate the practical meaning of the equation $f(20) = 255$

(b) What is the sign of $f'(t)$? Why?

(c) What are the units of $f'(20)$?

(d) In a complete sentence with units, what is the practical meaning of the statement $f'(20) = 2$?

Note: the units of the derivative of any function $y = f(x)$, will always be the y-axis units over the x-axis units. Remember, the derivative is essentially a division process, which is one reason many prefer Leibniz notation, $\frac{df}{dx}$.

Example 2:
An ice cream company knows that the cost, C (in dollars), to produce q quarts of cookie dough ice cream is a function (in part) of the number of quarts they produce, so $C = f(q)$.

(a) If $f(200) = 70$, explain in a full sentence with units exactly what this mathematical "sentence" is telling us in the context of the problem.

(b) If $f'(200) = 3$, explain in a full sentence with units exactly what this mathematical "sentence" is telling us in the context of the problem.

(c) If $f''(200) = -0.1$ explain in a full sentence with units exactly what this mathematical "sentence" is telling us in the context of the problem.

Example 3:
Let $f(t)$ be the amount of rain, in inches, fallen since midnight, where t is the time in hours since midnight. In a complete sentence including units, interpret the following in the context of the problem.

(a) $f(10) = 3.1$

(b) $f^{-1}(10) = 16$

(c) $f'(8) = 0.4$

(d) $\left(f^{-1}\right)'(5) = 2$

The word "**instantaneous**" is often used, even when x does not represent time. The word is actually often omitted. In practice, when we merely say "**rate of change**," at some moment in time, whether we reference a moment in time or some event, we mean "instantaneous rate of change."

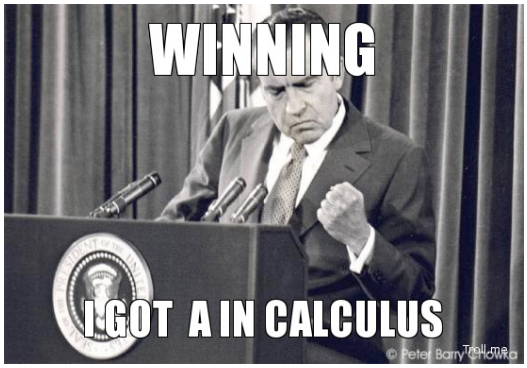
Example 4:
Economics: In the October 1996 issue of the *Notices of the AMS*, Hugo Rossi wrote, "In the fall of 1972 President Nixon announced that the rate of increase of inflation was decreasing. This was the first time a sitting president used the third derivative to advance his case for reelection."

(a) Laugh out Loud

(b) Explain what Rossi meant. Explain what Nixon meant.

Here's a quick summary from the definitive source of all knowledge, the modern Library of Alexandria, but with more fire protection—Wikipedia.

> Since inflation is itself a derivative—the rate at which the purchasing power of money decreases—then the rate of increase of inflation is the derivative of inflation, or the second derivative of the function of purchasing power of money with respect to time. Stating that a function is decreasing is equivalent to stating that its derivative is negative, so Nixon's statement is that the second derivative of inflation—or the third derivative of purchasing power—is negative.

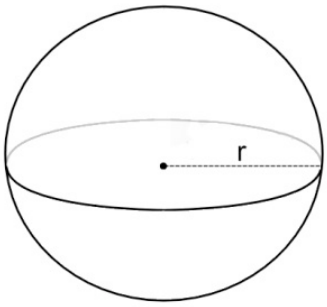


Orwellian Twist (Newspeak): Nixon's statement actually allowed for the rate of inflation to increase, so in actuality, his statement was not as indicative of stable prices as it sounds. Nonetheless, Nixon went on to win the general election over George McGovern with 60.7% of the popular vote.

Example 5:
Imagine a sphere whose radius is decreasing. When the diameter of the sphere is 6 feet, with respect to the radius

(a) How fast is the volume of the sphere changing? Explain your result.

(b) How fast is the surface area of the sphere changing? Explain your result.



(b) How fast is the diameter of the sphere changing? Explain your result.

Memorize:

Volume of a Sphere: $V = \frac{4}{3}\pi r^3$	Surface Area of a Sphere: $S = 4\pi r^2$	Shape of Sphere: Round

There is perhaps no more important, if not fun, type of rate of change than that of **motion**. After all, this was Newton's motivation for inventing Calculus to begin with. We have already looked at motion briefly. Here is a summary with some new terminology.

- $s''(t) = v'(t) = a(t)$, where $s(t)$ is the position function.
- Displacement over an interval $t \in [a,b]$: Displacement= $s(b) - s(a)$
- Distance traveled is the sum of the absolute values of the distances between turning points.
- Average velocity= $\dfrac{\text{displacement}}{\text{elapsed time}} = \dfrac{s(b)-s(a)}{b-a}$ = slope of secant line
- Velocity at time t: $s'(t) = v(t)$= slope of tangent line
- Speed= $|\text{velocity}| = |v(t)| = \left|\dfrac{ds}{dt}\right|$
- At time $t = c$, if both $v(c) > 0$ and $a(c) > 0$ OR if both $v(c) < 0$ and $a(c) < 0$, then speed is increasing.
- At time $t = c$, if $v(c) > 0$ and $a(c) < 0$ OR if $v(c) < 0$ and $a(c) > 0$, then speed decreasing.
- At time $t = c$, if $a(c) > 0$, velocity is increasing. If $a(c) < 0$, velocity is decreasing.
- At time $t = c$, if $v(c) > 0$, position (distance) is increasing and motion is in the positive direction (right or up).
- At time $t = c$, if $v(c) < 0$, position (distance) is decreasing and motion is negative direction (left or down).
- Direction changes when velocity changes from either positive to negative or from negative to positive at a point. Velocity need not pass through zero or even defined at the point where it's sign changes for direction to change.

Example 6:
A particle moves along a horizontal line so that its position at any time $t \geq 0$ is given by the function $x(t) = t^2 - 4t + 3$, where x is measured in feet and t is measured in seconds.

(a) Find the displacement of the particle during the first 3 seconds. Explain its meaning.

(b) Find the average velocity of the particle during the first 3 seconds. Explain its meaning.

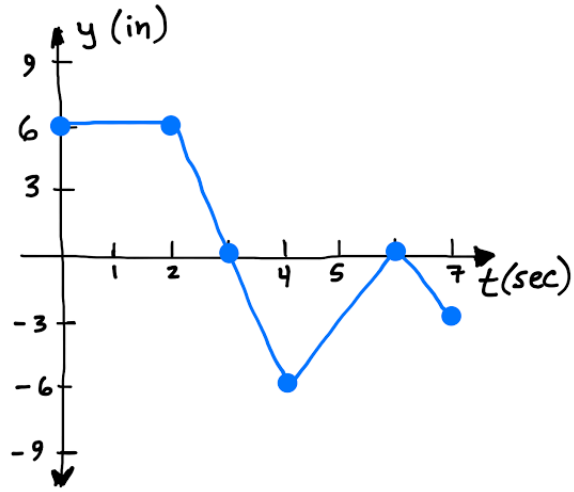
(c) Find the particle's initial velocity and its velocity at $t = 3$ seconds. Explain the meanings of each in terms of the particle's movement.

(d) Find the acceleration of the particle when $t = 3$ seconds. Explain its meaning in terms of the particle's velocity.

(e) At $t = 3$ seconds, is the speed of the particle increasing or decreasing? Justify.

(f) During what times is the particle moving to the right? Left? At what values of t does the particle change direction? Justify.

(g) Find the total distance the particle travels during the first 3 seconds. Are you as exhausted as the particle?



Example 7:

The graph above shows the position $y(t)$ of a particle, in inches, moving along a vertical line for $0 \le t \le 7$ seconds.

(a) On what open intervals $0 < t < 7$ is the particle moving down? Justify.

(b) On what open intervals or at what time(s) $0 < t < 7$ is the particle at rest? Justify.

(c) At what time(s) $0 < t < 7$ does the particle change direction? Justify.

(d) During the first 6 seconds, what is the particle's displacement? Explain what this means in terms of the particle's change in position.

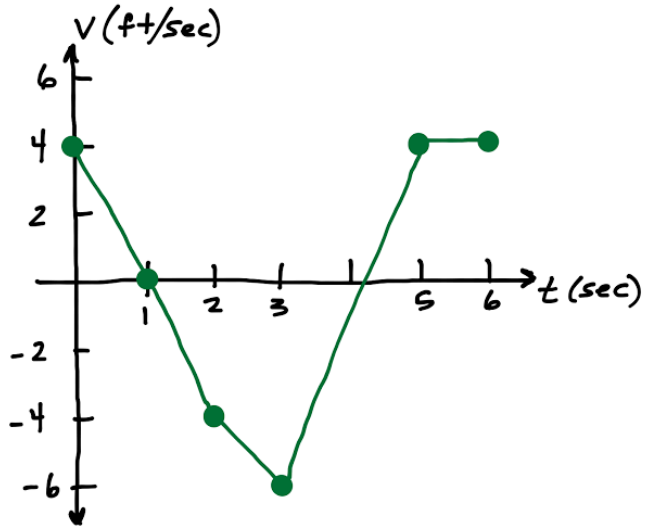
(e) During the first 6 seconds, through how many inches does the particle travel?

(f) What is the particle's velocity at $t = 5$ seconds? Explain what this means in terms of the particle's movement.

(g) What is the particle's velocity at $t = 5.9999$ seconds? $t = 6$ seconds?

(h) During what open time interval(s) $0 < t < 7$ is the particle's SPEED the greatest? VELOCITY the greatest??

(i) What is the particle's acceleration at $t = 3$ seconds?



Example 8:
The graph above shows the velocity $v(t)$ of a particle, in ft/sec, moving along a horizontal line for $0 \le t \le 6$ seconds.

(a) On what open intervals or at what time(s) $0 < t < 6$ is the particle at rest? Justify.

(b) On what open intervals $0 < t < 6$ is the particle moving to the right? Justify.

(c) On what open intervals or at what time(s) $0 < t < 6$ is the particle moving at its greatest speed? Greatest velocity?

(d) On what open intervals or at what time(s) $0 < t < 6$ is the particle's speed increasing? Decreasing? Justify.

(e) What is the particle's acceleration at $t = 4.8$ second? Explain what this number means in terms of the particle's velocity.

(f) On what open intervals or at what time(s) $0 < t < 6$ is the acceleration of the particle the greatest?

(g) (is for "genius") What is the particle's displacement during the 2 seconds? Justify.

Sometimes we aren't given an equation or a graph, but rather a table of discreet values. In these cases, we are at the mercy of the limited information given to us. Sometimes, too, we are not given units. This liberates us from having to include them in our final numeric answers and/or verbal explanations. We're still very much aware of the consequences, though, of possible units.

Example 9:
The values of the coordinate ***s*** of a bug moving smoothly and continuously along a line for various values of $t \in [0,4]$ are given below.

t	0	0.5	1	1.5	2	2.7	3	3.6	4
$s(t)$	40.0	35.0	30.2	36.0	48.2	45.0	38.2	16.0	0.2

(a) What is the displacement of the bug during the given interval $t \in [0,4]$? Show the work that leads to your answer.

(b) What is the minimum number of times the bug changes directions for $t \in [0,4]$? Explain your reasoning.

(c) What is the bug's average velocity for $t \in [0,0.5]$? Show the work that leads to your answer.

(d) Estimate the bug's velocity at each of the following. Use proper notation (always), and show the work that leads to your answers.

(i) At $t = 0.5$ (ii) At $t = 2.7$ (iii) At $t = 3.5$

(e) From the information given, is it possible to determine the time and position of the bug when it is farthest away from the origin? Why or why not?