Integration:  
By Parts & By Partial Fractions

Integration by parts is used to integrate a product, such as the product of an algebraic and a transcendental function:

\[ \int x e^x \, dx, \int x \sin x \, dx, \int x \ln x \, dx, \text{ etc. . . } \]

From the product rule for differentiation for two functions \( u \) and \( v \):

\[ \frac{d}{dx}(uv) = u'v + uv' = v \frac{du}{dx} + u \frac{dv}{dx} \]

If we integrate both sides and solve for \( \int uv \, dv \), we get the following:

\[ \int \left( \frac{d}{dx}(uv) \right) \, dx = \int \left( v \left( \frac{du}{dx} \right) + u \left( \frac{dv}{dx} \right) \right) \, dx \]

\[ uv = \int vdu + \int u \, dv \]

\[ \int u \, dv = uv - \int v \, du \]

This last line gives us a method for integrating by identifying two parts: a function we know how to differentiate (called \( u \)) and a function we know how to integrate (called \( dv \)).

Let's look at each of the examples above:

**Ex**: 1. \( \int x e^x \, dx \)

We first have to realize that we can integrate and differentiate both factors \( x \) and \( e^x \). When one of the functions is a polynomial, the rule of thumb is to let it be called \( u \), and to let the other factor be called \( dv \). We can set the problems up the same way every time:

\[ u = x \quad \text{calculate this} \quad \frac{du}{dx} = 1 \]

\[ dv = e^x \, dx \quad \text{calculate this} \quad v = e^x \]

Now we can follow a pattern from top right in a "backwards z" pattern:

\[ \int x e^x \, dx = xe^x - \int e^x \, dx \]
At this point, we have part of the final answer, but we still have an integral. Reevaluate this integral and integrate it by any appropriate method. You will find, however, that most of the time, this will be a routine integral, as is the case here. Continuing with the right side of the equations . . .

\[
\int xe^x\,dx = xe^x - \int e^x \, dx \\
= xe^x - e^x + C
\]

DONE!!! Go ahead and verify by differentiating it. You should get our original integrand.

Let’s do the next example more quickly (without as much commentary).

Ex: 2. \( \int x \sin x \, dx \)

\[
\begin{align*}
    u &= x & dv &= \sin x \, dx \\
    du &= dx & v &= -\cos x \\
\end{align*}
\]

so

\[
\int x \sin x \, dx = -x \cos x - \int (-\cos x) \, dx \\
= -x \cos x + \int \cos x \, dx \\
= -x \cos x + \sin x + C
\]

Ex: 3. \( \int x \ln x \, dx \)

If we follow our pattern above, we would let \( dv = \ln x \, dx \). This means we must know how to anti-differentiate \( \ln x \), which we don’t know how to do . . . . . . yet. In this case, our hand is forced—we **must** let \( u = \ln x \) and \( dv = x \, dx \).

\[
\begin{align*}
    u &= \ln x & dv &= x \, dx \\
    du &= \frac{1}{x} \, dx & v &= \frac{1}{2} x^2 \\
\end{align*}
\]

so

\[
\int x \ln x \, dx = \frac{1}{2} x^2 \ln x - \int \left( \frac{1}{2} x^2 \right) \left( \frac{1}{x} \right) \, dx \\
= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x \, dx \\
= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C
\]

A little longer, but still pretty darn nice.
Let's go ahead and see what the anti-derivative of plain ol' \( \ln x \) is.

**Ex: 4.** \( \int \ln x \, dx \)

Let \( u = \ln x \) and \( dv = dx \). Then \( du = \frac{1}{x} \, dx \) and \( v = x \). Thus:

\[
\int \ln x \, dx = x \ln x - \int (x) \left( \frac{1}{x} \right) \, dx
\]

So:

\[
\int \ln x \, dx = x \ln x - \int dx
\]

\[
= x \ln x - x + C
\]

Can it be this easy every time? Of course . . . . . . . . . . not. Sometimes we may have to apply this process more than once, maybe even more than twice!!!! Oh no!

**Ex: 5.** \( \int x^3 \cos x \, dx \)

Let \( u = x^3 \) and \( dv = \cos x \, dx \). Then \( du = 3x^2 \, dx \) and \( v = \sin x \). Thus:

\[
\int x^3 \cos x \, dx = x^3 \sin x - 3 \int x^2 \sin x \, dx
\]

At this point, we must repeat the process two more time on the remaining integrals, not forgetting the parts already integrated. When this is the case, the tabular method (popularized in the movie "Stand and Deliver" as the "tic-tac-toe" method) works very efficiently. Follow this pattern:

<table>
<thead>
<tr>
<th>SIGN (start with + &amp; alternate)</th>
<th>( u ) (go to zero if you can)</th>
<th>( dv )</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>( x^3 )</td>
<td>( \cos x )</td>
</tr>
<tr>
<td>-</td>
<td>( 3x^2 )</td>
<td>( \sin x )</td>
</tr>
<tr>
<td>+</td>
<td>( 6x )</td>
<td>( -\cos x )</td>
</tr>
<tr>
<td>-</td>
<td>( 6 )</td>
<td>( -\sin x )</td>
</tr>
<tr>
<td>+</td>
<td>( 0 )</td>
<td>( \cos x )</td>
</tr>
</tbody>
</table>

Following the arrows and writing down the products of the diagonals we get the following answer:

\[
\int x^3 \cos x \, dx = x^3 \sin x - 3x^2 (-\cos x) + 6x (-\sin x) - 6 \cos x \quad (\text{the next term would be zero})
\]

\[
= x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C \quad (\text{the } +C \text{ is the } "C"\text{herry on top})
\]
Partial fractions also works well for definite integrals, just find the antiderivative first, then evaluate the difference at the endpoints, just do it as two separate calculations with the proper notation.

Ex: 6. \( \int_{0}^{1} (7 - 3x)e^{6x} \, dx \)

\[
\int (7 - 3x)e^{6x} \, dx
\]

<table>
<thead>
<tr>
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<th>( u ) (go to zero if you can)</th>
<th>( dv )</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>( 7 - 3x )</td>
<td>( e^{6x} )</td>
</tr>
<tr>
<td>-</td>
<td>( -3 )</td>
<td>( \frac{1}{6} e^{6x} )</td>
</tr>
<tr>
<td>+</td>
<td>( 0 )</td>
<td>( \frac{1}{36} e^{6x} )</td>
</tr>
</tbody>
</table>

\[
\int (7 - 3x)e^{6x} \, dx = \frac{1}{6} (7 - 3x)e^{6x} - (-3) \left( \frac{1}{36} e^{6x} \right) + 0 + C
\]

\[
= \frac{1}{6} (7 - 3x)e^{6x} + \frac{1}{12} e^{6x} + C
\]

so

\[
\int_{0}^{1} (7 - 3x)e^{6x} \, dx = \frac{1}{6} (7 - 3x)e^{6x} + \frac{1}{12} e^{6x} \bigg|_{0}^{1} = \left[ \frac{2e^6}{3} + \frac{e^6}{12} \right] - \left[ \frac{7}{6} + \frac{1}{12} \right] = \frac{3e^6 - 5}{4}
\]

There are variations of integration by parts where the tabular method is additionally useful, among them are the cases when we have the product of two transcendental functions, such that the integrand “repeats” itself. Currently, this is not tested on the AP Calculus BC exam.

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**Integration by Partial Fractions**

Currently, College Board requires BC students to be able to integrate by the method of partial fractions for Linear, Non-Repeating factors only. Bear in mind that there are other, more involved partial fraction problems out there.

We will be using partial fractions when our denominator is factored into the product of linear, non-repeating factors (or can be written as such). It involves “splitting” up a single fraction into the sum or difference of multiple fractions (which can each be integrated separately).
This simple example illustrates this simple concept of adding fractions by getting a common denominator:

Since \( \frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6} \), we can say that a partial fractions decomposition for \( \frac{5}{6} \) is

\[
\frac{5}{6} = \frac{1}{2} + \frac{1}{3}
\]

Easy, huh? We all know how to add by getting a common denominator. The secret is how to decompose (going backwards) when we are not using numbers, but rather, variable expressions.

**Ex. 1** \( \int \frac{x-1}{x^2 + x} \) (this problem is almost a natural log \( u \)-substitution, but not quite)

**Step 1:** Factor the denominator

\[
\int \frac{x-1}{x^2 + x} \, dx = \int \frac{x-1}{x(x+1)} \, dx
\]

(notice the denominator now has two linear, non-repeating factors)

**Step 2:** Split up the denominators with variables, A & B as the temporary numerators.

\[
\int \frac{x-1}{x(x+1)} \, dx = \int \frac{A}{x} + \frac{B}{x+1} \, dx
\]

**Step 3:** Now the meat of the problem. (I will show the “cover up” method here, but the math behind it can be taught as a separate lesson.)

a. Take the value that makes the first term’s denominator equal to zero, in this case it is \( x = 0 \).

b. Now, on the left side of the equation, in the integrand, “cover up” the factor of \( x \) (the reason we get a zero) and plug our numeric value (\( x = 0 \)) into the rest of the expression. The result will be the numerator for the “covered up” term.

\[
\frac{x-1}{x(x+1)} \rightarrow \text{plug in } x = 0 \text{ for } x \Rightarrow \frac{0-1}{0+1} = -1 = A
\]

c. Now repeat the process for the second, third, fourth terms, etc. For our only other term, \( x = -1 \) make the denominator zero.

d. So “covering up” the \( x + 1 \) factor in the integrand and plugging in \( x = -1 \):

\[
\frac{x-1}{x(x+1)} \rightarrow \text{plug in } x = 0 \text{ for } x \Rightarrow \frac{-1-1}{-1} = 2 = B
\]

e. Now rewrite your integral with the values of \( A \) and \( B \) found in parts b) and d):

\[
\int \frac{x-1}{x(x+1)} \, dx = \int \frac{A}{x} + \frac{B}{x+1} \, dx = \int -\frac{1}{x} + \frac{2}{x+1} \, dx
\]
Integrate each term independently:

\[
\int \frac{x-1}{x^2 + x} \, dx = \int \left( -\frac{1}{x} + \frac{2}{x+1} \right) \, dx = -\ln|x| + 2\ln|x+1| + C
\]

Presto!

That’s all there is to partial fractions with linear, non-repeating factors. You might come across some with more than two distinct linear factors, but don’t fear, just proceed in the same fashion, and have fun!

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**WORSHEET ON INTEGRATION BY PARTS AND PARTIAL FRACTIONS**

Evaluate the following integrals without a calculator. Show all steps and substitutions.

1. \( \int xe^{3x} \, dx = \)

2. \( \int x^2 e^{3x} \, dx = \)

3. \( \int x^3 \ln x \, dx = \)

4. \( \int_0^{\pi/2} x \sin x \, dx = \)

5. \( \int \frac{4x + 41}{x^2 + 3x - 10} \, dx = \)

6. \( \int \frac{6}{x^2 - 1} \, dx = \)

Sometimes it is necessary to do long division before we decompose and integrand into partial fractions, this is required when the degree of the numerator is \( \geq \) the degree of the denominator.

7. \( \int \frac{4x^3 - 3x + 5}{x^2 - 2x} \, dx = \)
WORKSHEET Continued.

From the 1998 BC Multiple Choice
8. \[ \int \frac{1}{x^2 - 6x + 8} \, dx = \]

(A) \( \frac{1}{2} \ln \left| \frac{x - 4}{x - 2} \right| + C \) \hspace{1cm} (B) \( \frac{1}{2} \ln \left| \frac{x - 2}{x - 4} \right| + C \) \hspace{1cm} (C) \( \frac{1}{2} \ln |(x - 2)(x - 4)| + C \)

(D) \( \frac{1}{2} \ln |(x - 4)(x + 2)| + C \) \hspace{1cm} (E) \( \ln |(x - 2)(x - 4)| + C \)

9. \( \int x \cos x \, dx = \)

(A) \( x \sin x - \cos x + C \) \hspace{1cm} (B) \( x \sin x + \cos x + C \) \hspace{1cm} (C) \(-x \sin x + \cos x + C \)

(D) \( x \sin x + C \) \hspace{1cm} (E) \( \frac{1}{2} x^2 \sin x + C \)

From the 2003 BC Multiple Choice
10. \( \int x \sin(6x) \, dx = \)

(A) \(-x \cos(6x) + \sin(6x) + C \) \hspace{1cm} (B) \(- \frac{x}{6} \cos(6x) + \frac{1}{36} \sin(6x) + C \)

(C) \(- \frac{x}{6} \cos(6x) + \frac{1}{6} \sin(6x) + C \) \hspace{1cm} (D) \( \frac{x}{6} \cos(6x) + \frac{1}{36} \sin(6x) + C \)

(E) \( 6x \cos(6x) - \sin(6x) + C \)

11. \( \int \frac{2x}{(x + 2)(x + 1)} \, dx = \)

(A) \( \ln |x + 2| + \ln |x + 1| + C \) \hspace{1cm} (B) \( \ln |x + 2| + \ln |x + 1| - 3x + C \)

(C) \(-4 \ln |x + 2| + 2 \ln |x + 1| + C \) \hspace{1cm} (D) \( 4 \ln |x + 2| - 2 \ln |x + 1| + C \)

(E) \( 2 \ln |x| + \frac{2}{3} x + \frac{1}{2} x^2 + C \)